Linear differential equations

The solution of $\mathrm{d}x(t)/\mathrm{d}t = ax(t)$ is $x(t) = C\mathrm{e}^{at}$, where $C = x(0)$.

Check this:

$$\partial_t \, C\mathrm{e}^{at} = aC\mathrm{e}^{at} = ax(t)$$

Now two-dimensional systems:

$$\begin{cases} \mathrm{d}x/\mathrm{d}t = f(x, y) \\ \mathrm{d}y/\mathrm{d}t = g(x, y) \end{cases}$$

where $x(t)$ and $y(t)$ are unknown functions of time t, and f and g are functions of x and y.

An example:

$$\begin{cases} \mathrm{d}x/\mathrm{d}t = ax + by \\ \mathrm{d}y/\mathrm{d}t = cx + dy \end{cases} \quad \text{and} \quad \begin{cases} \mathrm{d}x/\mathrm{d}t = -2x + y \\ \mathrm{d}y/\mathrm{d}t = x - 2y \end{cases}$$

where x and y decay at a rate -1, and are converted into one another at a rate 1. Steady state $x = y = 0$.

In matrix notation:

$$\begin{pmatrix} \mathrm{d}x/\mathrm{d}t \\ \mathrm{d}y/\mathrm{d}t \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

We claim that $\begin{pmatrix} \mathrm{d}x/\mathrm{d}t \\ \mathrm{d}y/\mathrm{d}t \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ has as a general solution:

$$\begin{aligned} x(t) &= C_1 x_1 \mathrm{e}^{\lambda_1 t} + C_2 x_2 \mathrm{e}^{\lambda_2 t} \\ y(t) &= C_1 y_1 \mathrm{e}^{\lambda_1 t} + C_2 y_2 \mathrm{e}^{\lambda_2 t} \end{aligned}$$

or

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \mathrm{e}^{\lambda_1 t} + C_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \mathrm{e}^{\lambda_2 t}$$

where $\lambda_{1,2}$ are eigenvalues and $(x_i \; y_i)$ are the corresponding eigenvectors of the matrix given above.

Like $x(t) = C\mathrm{e}^{at}$, this has only one steady state: $(x, y) = (0, 0)$.

Notice that the solutions $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \mathrm{e}^{\lambda_1 t} + C_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \mathrm{e}^{\lambda_2 t}$ are a linear combination of the growth along the eigenvectors.

Since $x(t)$ and $y(t)$ grow when $\lambda_{1,2} > 0$ we obtain:

- a stable node when both $\lambda_{1,2} < 0$
- an unstable node when both $\lambda_{1,2} > 0$
- an (unstable) saddle point when $\lambda_1 > 0$ and $\lambda_2 < 0$ (or vice versa)

When $\lambda_{1,2}$ are complex, i.e., $\lambda_{1,2} = \alpha \pm \mathrm{i}\beta$, we obtain

- a stable spiral when the real part $\alpha < 0$
- an unstable spiral when the real part $\alpha > 0$
- a neutrally stable center point when the real part $\alpha = 0$

Example: $\begin{pmatrix} \mathrm{d}x/\mathrm{d}t \\ \mathrm{d}y/\mathrm{d}t \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Since tr $= -4$ and det $= 4 - 1 = 3$ we obtain:

$$\lambda_{1,2} = \frac{-4 \pm \sqrt{16 - 12}}{2} = -2 \pm 1$$

so $\lambda_1 = -1$ and $\lambda_2 = -3$.

Hence solutions tend to zero and $(x, y) = (0, 0)$ is a stable node.

To find the eigenvector $\vec{v_1}$ we write:

$$\vec{v_1} = \begin{pmatrix} -b \\ a - \lambda_1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad \text{or} \quad \vec{v_1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\vec{v_2}$ we write

$$\vec{v_2} = \begin{pmatrix} -b \\ a - \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

In combination this gives

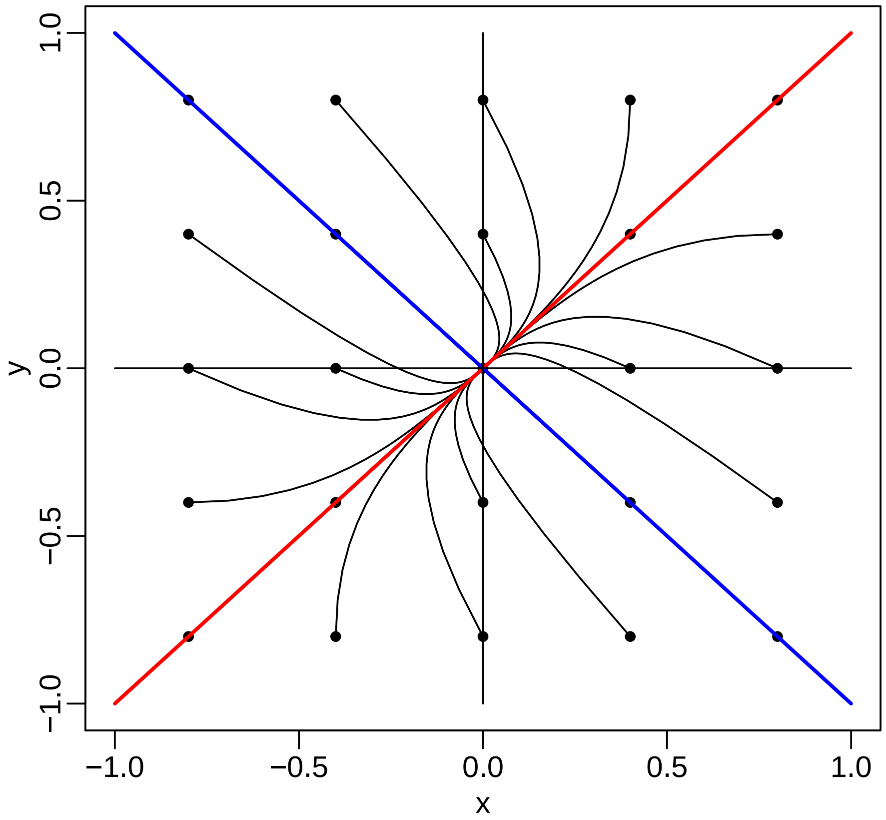
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{e}^{-t} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \mathrm{e}^{-3t}$$

or

$$\begin{aligned} x(t) &= C_1 \mathrm{e}^{-t} - C_2 \mathrm{e}^{-3t} \\ y(t) &= C_1 \mathrm{e}^{-t} + C_2 \mathrm{e}^{-3t} \end{aligned}$$

The integration constants C_1 and C_2 can be solved from the initial condition: i.e., $x(0) = C_1 - C_2$ and $y(0) = C_1 + C_2$.

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{e}^{-t} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \mathrm{e}^{-3t}$$



Finally let's check this solution:

$$
\begin{aligned}
x(t) &= C_1 \mathrm{e}^{-t} - C_2 \mathrm{e}^{-3t} \\
y(t) &= C_1 \mathrm{e}^{-t} + C_2 \mathrm{e}^{-3t}
\end{aligned}
$$

or

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= -C_1 \mathrm{e}^{-t} + 3C_2 \mathrm{e}^{-3t} \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= -C_1 \mathrm{e}^{-t} - 3C_2 \mathrm{e}^{-3t}
\end{aligned}
$$

which should be equal to

$$
\frac{\mathrm{d}x}{\mathrm{d}t} = -2x+y = -2(C_1 \mathrm{e}^{-t} - C_2 \mathrm{e}^{-3t}) + C_1 \mathrm{e}^{-t} + C_2 \mathrm{e}^{-3t} = -C_1 \mathrm{e}^{-t} + 3C_2 \mathrm{e}^{-3t}
$$

$$
\frac{\mathrm{d}y}{\mathrm{d}t} = x-2y = C_1 \mathrm{e}^{-t} - C_2 \mathrm{e}^{-3t} - 2(C_1 \mathrm{e}^{-t} + C_2 \mathrm{e}^{-3t}) = -C_1 \mathrm{e}^{-t} - 3C_2 \mathrm{e}^{-3t}
$$

Linear approximations

Derivative:

$$f'(\bar{x}) = \lim_{x \to \bar{x}} \frac{f(x) - f(\bar{x})}{x - \bar{x}} \quad \text{or} \quad f'(\bar{x}) = \lim_{h \to 0} \frac{f(\bar{x} + h) - f(\bar{x})}{h} \ ,$$

Rewrite this into:

$$f(x) \simeq f(\bar{x}) + f'(\bar{x})\,(x - \bar{x}) \quad \text{or} \quad f(x) \simeq f(\bar{x}) + f'(\bar{x})\,h \ ,$$

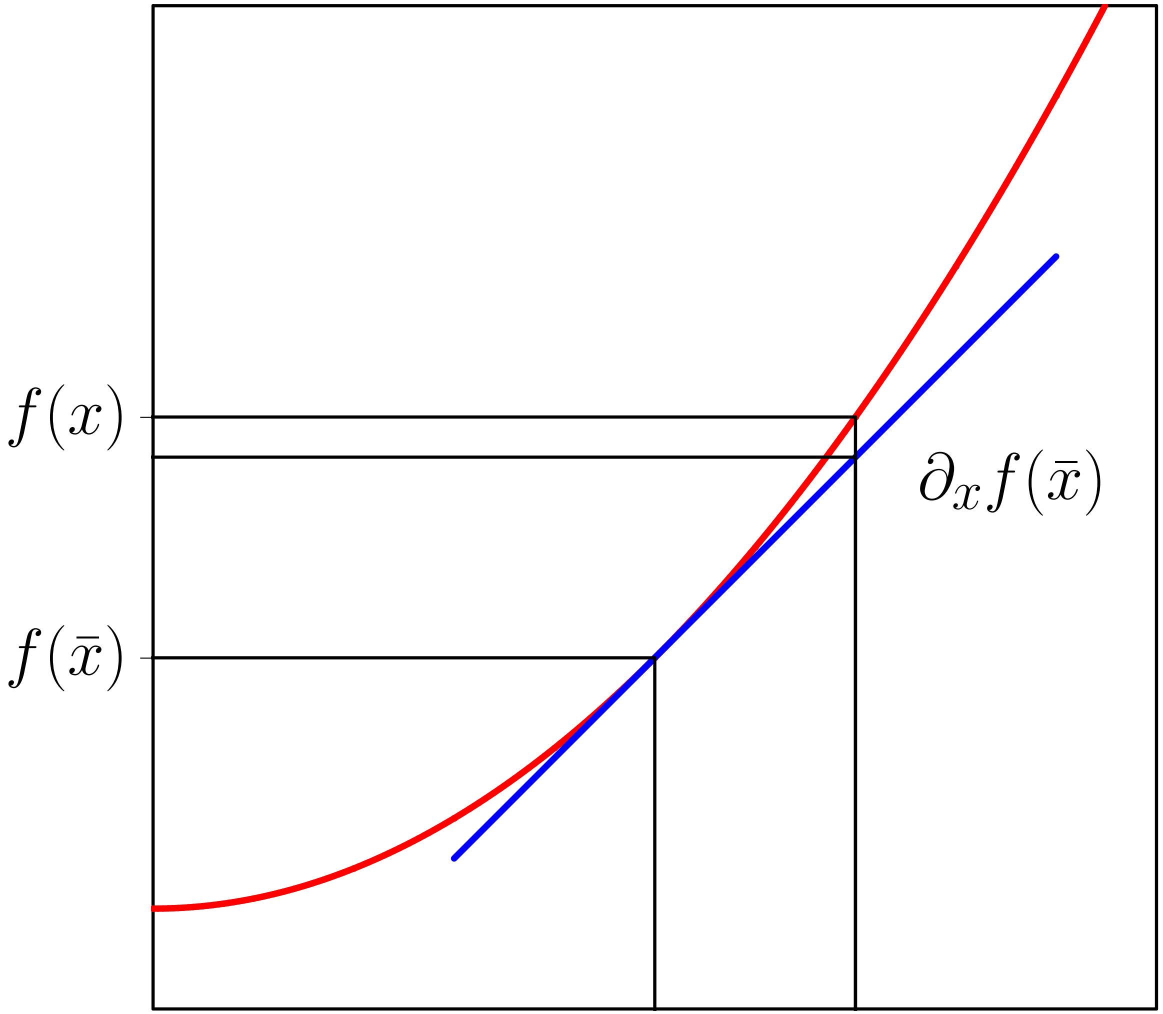
Example:

$$f(x) = ax^2 + b \quad \to \quad \partial_x f(x) = 2ax$$

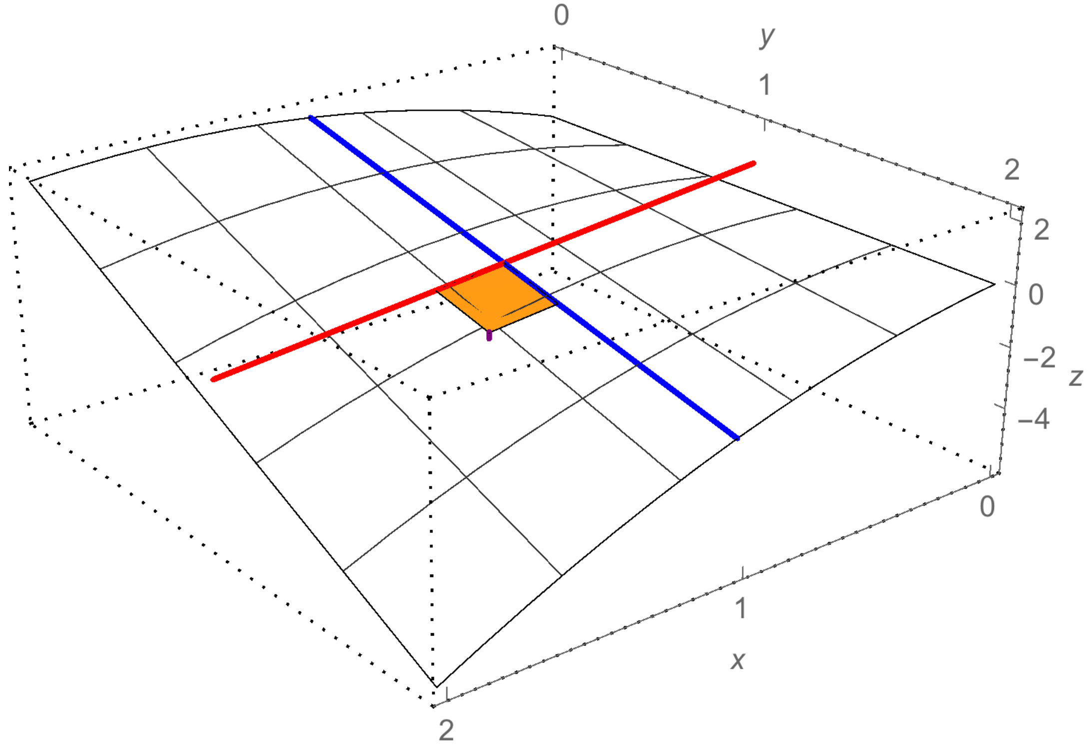
$$a = 2\,, b = 1\,, x = 3 \ \to \ f(3) = 2{\times}9{+}1 = 19\,, \partial_x f(3) = 2{\times}2{\times}3 = 12$$

$$f(3.1) = 20.22 \quad \text{or} \quad f(3.1) \simeq f(3){+}\partial_x f(3){\times}0.1 = 19{+}12{\times}0.1 = 20.2$$

$$f(x) \simeq f(\bar{x}) + \partial_x f(\bar{x})\,(x - \bar{x})$$



The function $f(x, y) = 3x - x^2 - 2xy$:



$$\partial_x f(x, y) = 3 - 2x - 2y \quad \text{and} \quad \partial_y f(x, y) = -2x$$

and in the point $f(1, 1) = 0$:

$$\partial_x\ f(x, y) = -1 \quad \text{and} \quad \partial_y\ f(x, y) = -2$$

Generally

$$f(x,y) \simeq f(\bar{x},\bar{y}) + \partial_x f \ (x-\bar{x}) + \partial_y f \ (y-\bar{y})$$

Or, after defining $h_x = x - \bar{x}$ and $h_y = y - \bar{y}$:

$$f(x,y) = f(\bar{x}+h_x, \bar{y}+h_y) \simeq f(\bar{x},\bar{y}) + \partial_x f \ h_x + \partial_y f \ h_y$$

Example:

$$f(x,y) = 3x - x^2 - 2xy \ , \quad f(1,1) = 0 \ , \quad \partial_x = -1, \partial_y = -2$$

$$f(1.25, 1.25) = 3.75 - 1.5625 - 3.125 = -0.9375$$

$$f(1.25, 1.25) \simeq 0 - 1 \times 0.25 - 2 \times 0.25 = -0.75$$

Consider

$$\begin{cases} \mathrm{d}x/\mathrm{d}t = f(x, y) \\ \mathrm{d}y/\mathrm{d}t = g(x, y) \end{cases}$$

close an equilibrium point at $(\bar{x}, \bar{y})$, i.e., $f(\bar{x}, \bar{y}) = g(\bar{x}, \bar{y}) = 0$

Linear approximation of $f(x, y)$ close to the equilibrium:

$$f(x, y) \simeq f(\bar{x}, \bar{y}) + \partial_x f \ (x - \bar{x}) + \partial_y f \ (y - \bar{y})$$

As $f(\bar{x}, \bar{y}) = 0$ we obtain

$$f(x, y) \simeq \partial_x f \ (x - \bar{x}) + \partial_y f \ (y - \bar{y})$$

For $g(x, y)$ this yields:

$$g(x, y) \simeq \partial_x g \ (x - \bar{x}) + \partial_y g \ (y - \bar{y})$$

$$\begin{cases} \mathrm{d}x/\mathrm{d}t = f(x,y) \\ \mathrm{d}y/\mathrm{d}t = g(x,y) \end{cases} \quad \text{became} \quad \begin{cases} \mathrm{d}x/\mathrm{d}t \simeq \partial_x f\ (x-\bar{x}) + \partial_y f\ (y-\bar{y}) \\ \mathrm{d}y/\mathrm{d}t \simeq \partial_x g\ (x-\bar{x}) + \partial_y g\ (y-\bar{y}) \end{cases}$$

Since the partial derivatives are merely the slopes of $f(x,y)$ and $g(x,y)$ at the point $(\bar{x},\bar{y})$, they are constants that we can write as

$$a = \partial_x f, \quad b = \partial_y f, \quad c = \partial_x g, \quad d = \partial_y g$$

Steady states $\bar{x}$ and $\bar{y}$ are also constants, with derivatives zero:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}\bar{x}}{\mathrm{d}t} = \frac{\mathrm{d}(x-\bar{x})}{\mathrm{d}t} \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t} - \frac{\mathrm{d}\bar{y}}{\mathrm{d}t} = \frac{\mathrm{d}(y-\bar{y})}{\mathrm{d}t}$$

Hence

$$\begin{cases} \mathrm{d}(x-\bar{x})/\mathrm{d}t = a(x-\bar{x}) + b(y-\bar{y}) \\ \mathrm{d}(y-\bar{y})/\mathrm{d}t = c(x-\bar{x}) + d(y-\bar{y}) \end{cases}$$

Changing variables to the distances $h_x = x - \bar{x}$ and $h_y = y - \bar{y}$:

$$\begin{cases} \mathrm{d}h_x/\mathrm{d}t = ah_x + bh_y \\ \mathrm{d}h_y/\mathrm{d}t = ch_x + dh_y \end{cases}$$

having the solution

$$\begin{pmatrix} h_x(t) \\ h_y(t) \end{pmatrix} = C_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \mathrm{e}^{\lambda_1 t} + C_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \mathrm{e}^{\lambda_2 t}$$

where $\lambda_{1,2}$ and $(x_i\ y_i)$ are the eigenvalues and corresponding eigenvectors of the Jacobi matrix

$$J = \begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Knowing the two eigenvalues of

$$J = \begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the steady state will be stable when $\lambda_1 < 0$ and $\lambda_2 < 0$.

If so the return time is defined by the largest eigenvalue:

$$T_R = \frac{-1}{\max(\lambda_1, \lambda_2)}$$

Example:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,y) = a - bx - cxy \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = g(x,y) = dxy - ey \ ,$$

with $\bar{x} = \frac{a}{b}$ when $y = 0$, and $\bar{x} = \frac{e}{d}$ and $\bar{y} = \frac{ad}{ce} - \frac{b}{c}$

$$J = \begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix} = \begin{pmatrix} -b - c\bar{y} & -c\bar{x} \\ d\bar{y} & d\bar{x} - e \end{pmatrix}$$

$$J = \begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix} = \begin{pmatrix} -b - c\bar{y} & -c\bar{x} \\ d\bar{y} & d\bar{x} - e \end{pmatrix}$$

Fill in $\bar{x} = \frac{a}{b}$ and $\bar{y} = 0$,

$$J_1 = \begin{pmatrix} -b & -\frac{ca}{b} \\ 0 & \frac{da}{b} - e \end{pmatrix}$$

Since this matrix is in a diagonal form we know that the diagonal elements provide the eigenvalues, i.e., $\lambda_1 = -b$ and $\lambda_2 = \frac{da}{b} - e$.

Hence this state is stable whenever $\lambda_2 < 0$, i.e., $\frac{a}{b} < \frac{e}{d}$.

$$J = \begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix} = \begin{pmatrix} -b - c\bar{y} & -c\bar{x} \\ d\bar{y} & d\bar{x} - e \end{pmatrix}$$

Now consider $\bar{x} = \frac{e}{d}$ and $\bar{y} = \frac{ad}{ce} - \frac{b}{c}$ and first fill in $\bar{x}$:

$$J_2 = \begin{pmatrix} -b - c\bar{y} & -\frac{ce}{d} \\ d\bar{y} & 0 \end{pmatrix}$$

When $\bar{y} > 0$ the signs of this matrix are given by

$$J_3 = \begin{pmatrix} -\alpha & -\beta \\ \gamma & 0 \end{pmatrix} \quad \text{with} \quad \mathrm{tr} J_3 = -\alpha < 0 \quad \text{and} \quad \det J_3 = \beta\gamma > 0\,,$$

such that

$$\lambda_{1,2} = \frac{\mathrm{tr} \pm \sqrt{\mathrm{tr}^2 - 4\det}}{2} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta\gamma}}{2} < 0\ ,$$

Since $\lambda_{1,2} < 0$ the non-trivial steady state is stable.

Having

$$J = \begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we know that

$$\lambda_{1,2} = \frac{\mathrm{tr} \pm \sqrt{D}}{2} \quad \text{where} \quad D = \mathrm{tr}^2 - 4\,\mathrm{det}$$

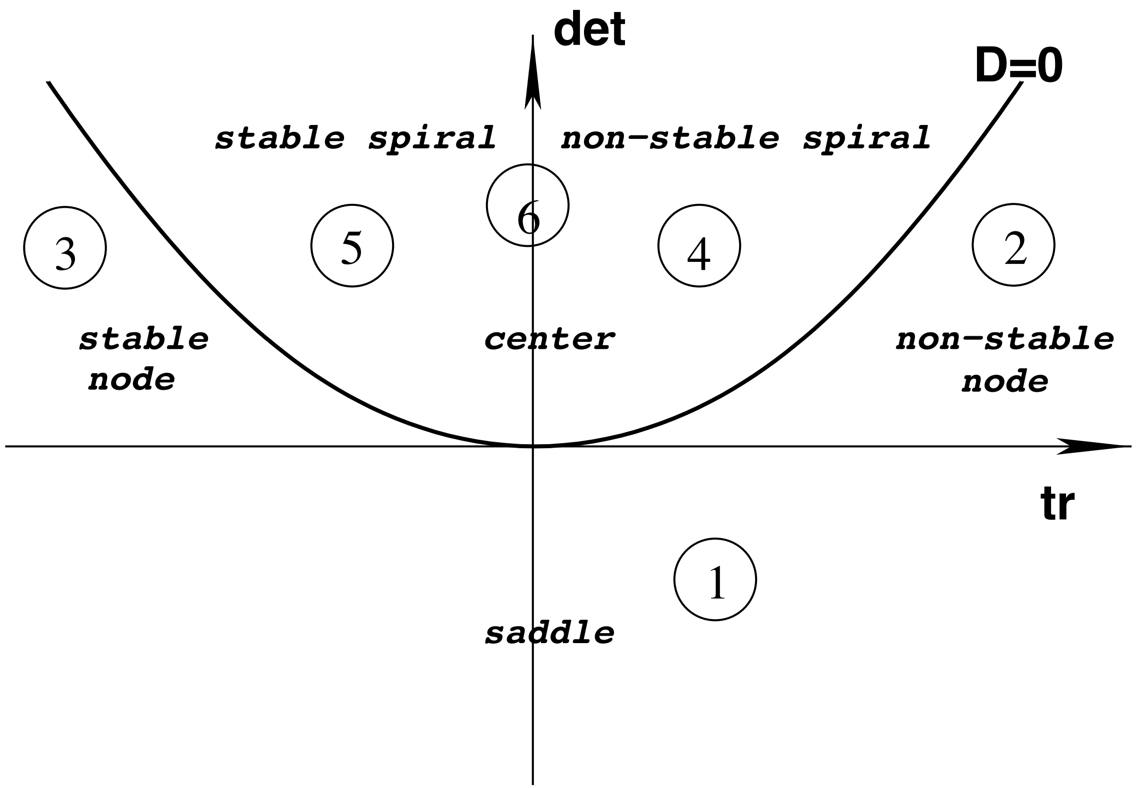
Observing that

$$\lambda_1 + \lambda_2 = \mathrm{tr}[J] \quad \text{and} \quad \lambda_1 \times \lambda_2 = \mathrm{det}[J]\ ,$$

the latter because

$$\frac{1}{4}(\mathrm{tr} + \sqrt{D})(\mathrm{tr} - \sqrt{D}) = \frac{1}{4}(\mathrm{tr}^2 - D) = \frac{1}{4}(\mathrm{tr}^2 - \mathrm{tr}^2 + 4\,\mathrm{det}) = \mathrm{det}$$

we can classify steady states by just the trace and determinant of their Jacobi matrix.



$$\lambda_{1,2} = \frac{\mathrm{tr} \pm \sqrt{D}}{2}$$

$$D = \mathrm{tr}^2 - 4\det$$

$$\lambda_1 + \lambda_2 = \mathrm{tr}$$

$$\lambda_1 \times \lambda_2 = \det$$

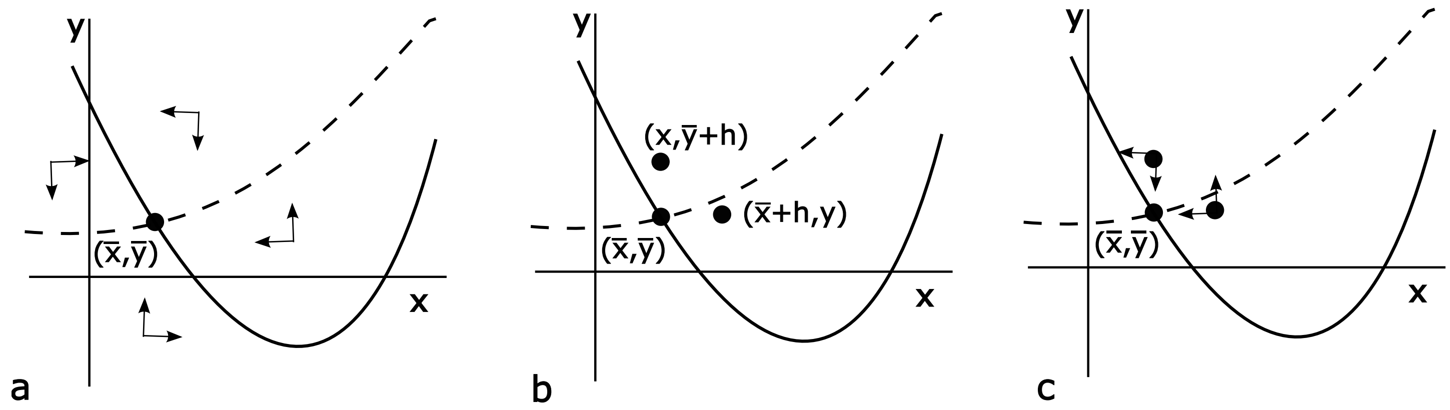
1. if $\det < 0$ then $D > 0$: $\lambda_{1,2}$ are real with unequal sign: saddle
2. if $\det > 0$, $\mathrm{tr} > 0$ and $D > 0$ then $\lambda_{1,2} > 0$: unstable node.
3. if $\det > 0$, $\mathrm{tr} < 0$ and $D > 0$ then $\lambda_{1,2} < 0$: stable node.
4. if $\det > 0$, $\mathrm{tr} > 0$ and $D < 0$ then $\lambda_{1,2} > 0$: unstable spiral.
5. if $\det > 0$, $\mathrm{tr} < 0$ and $D < 0$ then $\lambda_{1,2} > 0$: stable spiral.

Graphical Jacobian: use the signs only

$$J = \begin{pmatrix} \partial_x f \simeq \dfrac{f(\bar{x}+h,\bar{y})}{h} & \partial_y f \simeq \dfrac{f(\bar{x},\bar{y}+h)}{h} \\ \partial_x g \simeq \dfrac{g(\bar{x}+h,\bar{y})}{h} & \partial_y g \simeq \dfrac{g(\bar{x},\bar{y}+h)}{h} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with $\mathrm{tr}[J] = \alpha + \delta$ and $\det[J] = \alpha\delta - \beta\gamma$.

If tr < 0 and det > 0 the state will be stable.



$$J = \begin{pmatrix} \partial_x f \simeq \dfrac{f(\bar{x}+h,\bar{y})}{h} & \partial_y f \simeq \dfrac{f(\bar{x},\bar{y}+h)}{h} \\ \partial_x g \simeq \dfrac{g(\bar{x}+h,\bar{y})}{h} & \partial_y g \simeq \dfrac{g(\bar{x},\bar{y}+h)}{h} \end{pmatrix} = \begin{pmatrix} - & - \\ + & - \end{pmatrix}$$