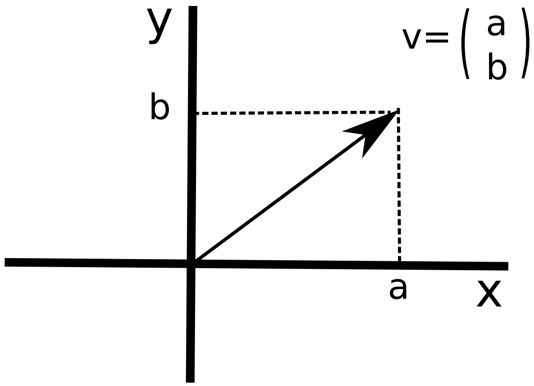
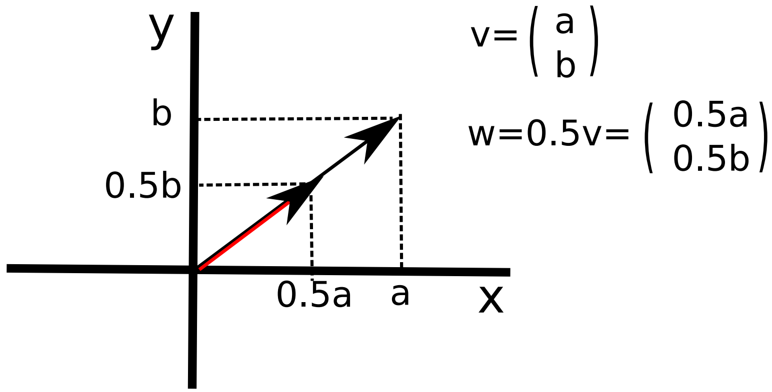
Vectors, matrices, eigenvalues and eigenvectors

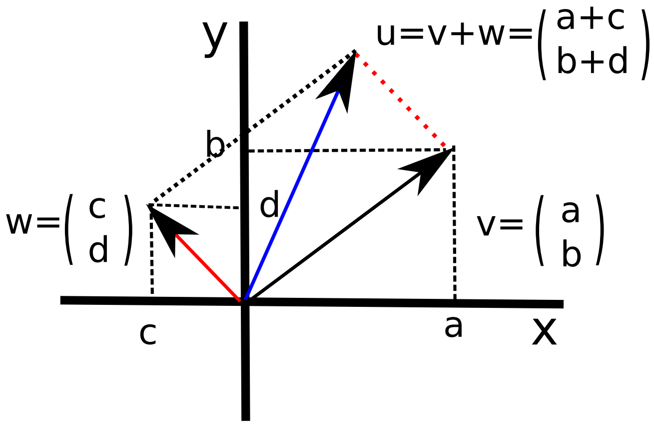
A vector in 2 dimensions



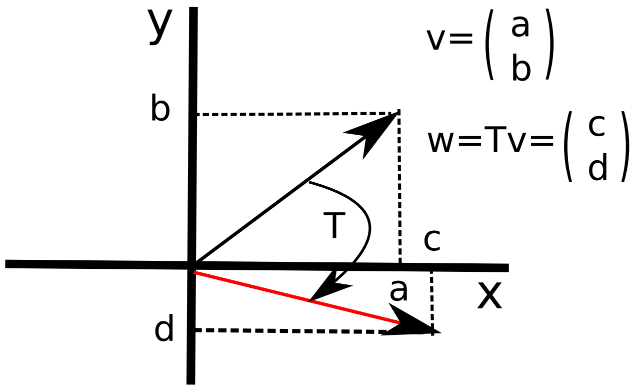
Scaling of vector



Adding of 2 vectors



Rotation of a vector



Scaling a vector: $0.5\vec{V} = 0.5\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.5 \times 2 \\ 0.5 \times 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}$

Adding two vectors: $\vec{V} + \vec{W} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2+1 \\ 1+3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

A scalar times a matrix: $\lambda\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}$

A matrix plus a matrix: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} a+x & b+y \\ c+z & d+w \end{pmatrix}$

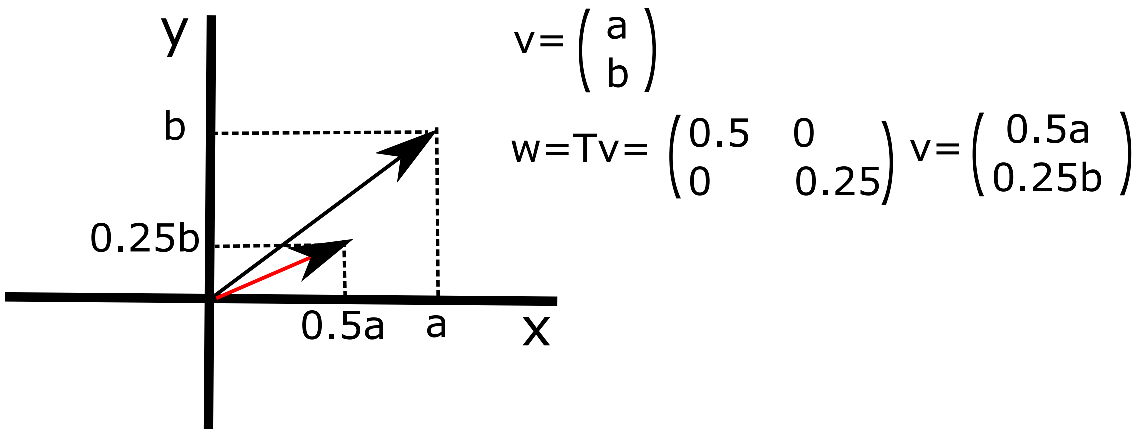
A matrix times a matrix: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{pmatrix}$

Hence the product of a matrix times a vector:

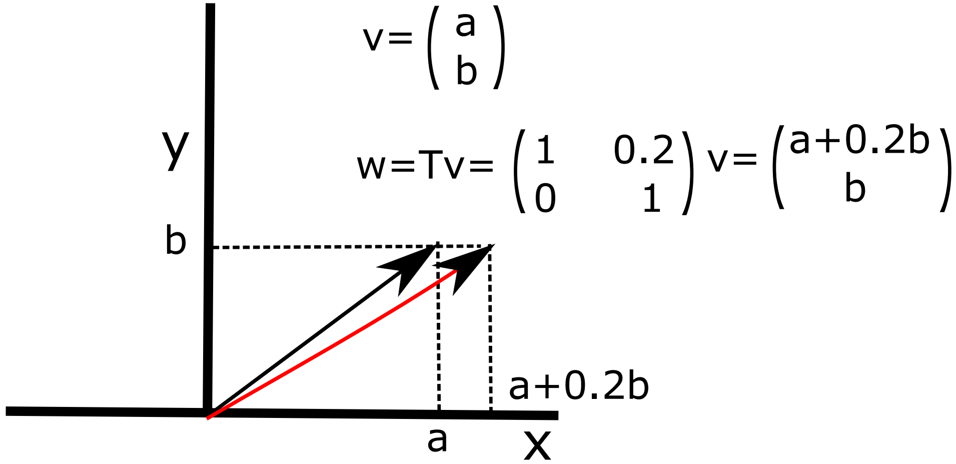
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

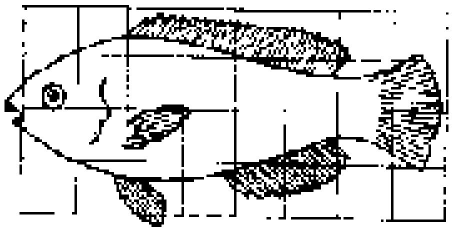
This matrix transforms the vector into another vector:

Complex scaling of vector

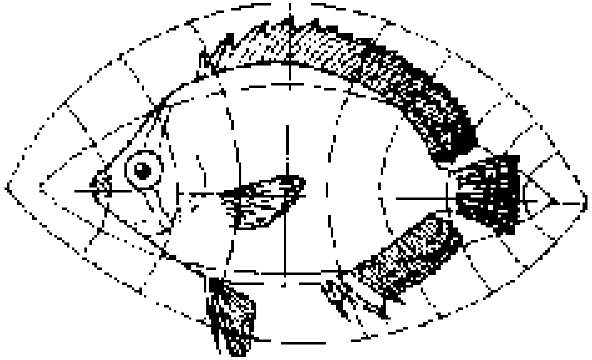


Shearing of vector parallel to x-axis

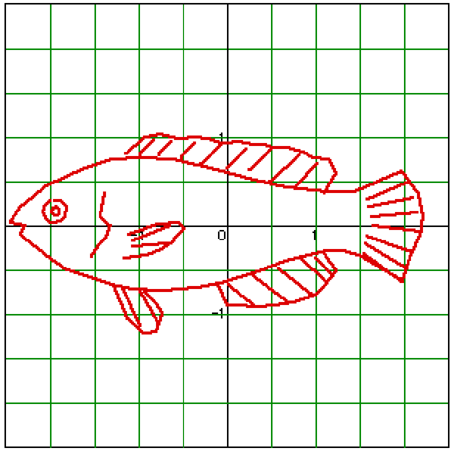


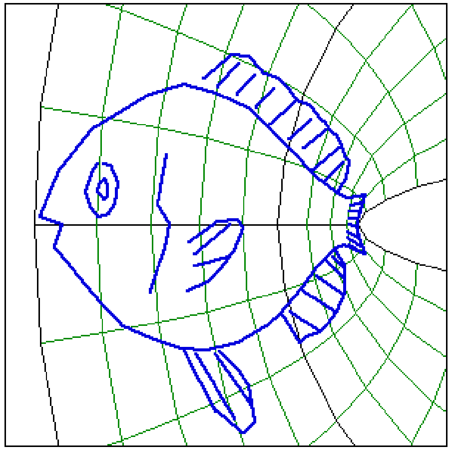


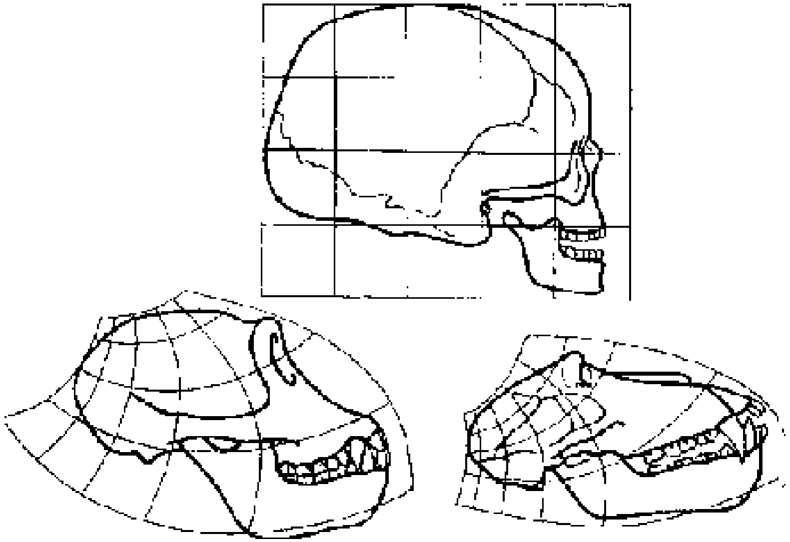
Scarus sp.



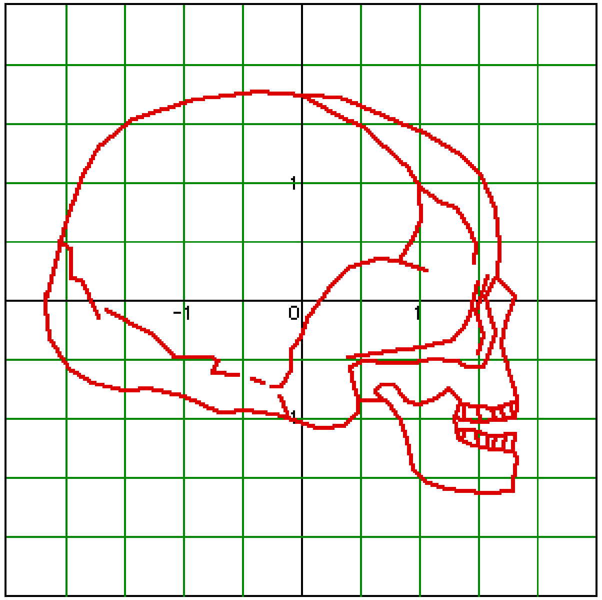
Pomacanthus.

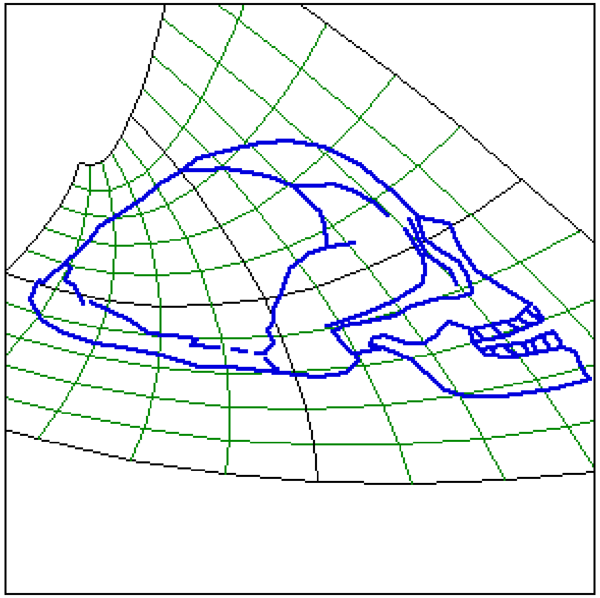






Skulls of a human, a chimpanzee and a baboon
and transformations between them





A system of linear equations: $\begin{cases} x - 2y = -5 \\ 2x + y = 10 \end{cases}$

Can be written as $A\vec{X} = \vec{V}$:

$$\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ 10 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

Which vector $\vec{X}$ is transformed by the matrix A into the vector $\vec{V}$?

If we were to know the inverse, A^{-1}, of the matrix A one could write

$$A\vec{X} = \vec{V} \quad \leftrightarrow \quad A^{-1}A\vec{X} = A^{-1}\vec{V} \quad \leftrightarrow \quad \vec{X} = A^{-1}\vec{V}$$

where A^{-1} is formally defined by the relationship, $A^{-1}A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -5 \\ 10 \end{pmatrix}$$

The inverse of $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

requiring that $ad - bc \neq 0$.

Check by matrix multiplication

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & -ab + ba \\ cd - dc & -cb + da \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}$$

$$= ad - bc \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence

$$\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}^{-1} = \frac{1}{1 + 4} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

and

$$\vec{X} = \frac{1}{1+4} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} -5 \\ 10 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 15 \\ 20 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Simplify solving linear equations by using the inverse matrix.

Previously, starting from $\begin{cases} x - 2y = -5 \\ 2x + y = 10 \end{cases}$ one would have solved the first equation to obtain $x = 2y - 5$, which gives in the second:

$2(2y-5)+y = 10 \quad$ or $\quad 4y-10+y = 10 \quad$ or $\quad 5y = 20 \quad$ i.e. $\quad y = 4\,$,

and hence $x = 3$.

Because the system can only solved when $ad - bc \neq 0$ we define the determinant and the trace of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as:

$$\det[A] = ad - bc \quad \text{and} \quad \mathrm{tr}[A] = a + d$$

Linear systems only have solutions when $\det[A] \neq 0$.

The importance of the trace wil appear later.

Things become more complicated for large matrices, e.g., for

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} , \quad \det[A] = aei - ceg + bfg - bdi + cdh - afh$$

Forest succession:

	Gray Birch	Blackgum	Red Maple	Beech
Gray Birch	0.05	0.01	0	0
Blackgum	0.36	0.57	0.14	0.01
Red Maple	0.5	0.25	0.55	0.03
Beech	0.09	0.17	0.31	0.96

For example, the fraction of Red Maple trees after 50 years would be $0.5\times$ the fraction of Gray Birch trees, plus $0.25\times$ the fraction of Blackgum trees, plus $0.55\times$ the fraction of Red Maples, plus $0.03\times$ the fraction of Beech trees.

Write table as a matrix:

$$A = \begin{pmatrix} 0.05 & 0.01 & 0 & 0 \\ 0.36 & 0.57 & 0.14 & 0.01 \\ 0.5 & 0.25 & 0.55 & 0.03 \\ 0.09 & 0.17 & 0.31 & 0.96 \end{pmatrix}$$

and define the current state of the forest as a vector, e.g.,

$$\vec{V_0} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$$

After 50 years the next state of the forest is defined by:

$$\vec{V_{50}} = A\vec{V_0} = \begin{pmatrix} 0.05 & 0.36 & 0.5 & 0.09 \end{pmatrix}$$

which is a forest with 5% Gray Birch, 36% Blackgum, 50% Red Maple, and 9% Beech trees.

The next state of the forest is

$$\vec{V_{100}} = A\vec{V_{50}} = \begin{pmatrix} 0.0061 & 0.2941 & 0.3927 & 0.3071 \end{pmatrix}$$

After 100 intervals of 50 years, the state is $\vec{V_{5000}} = A^{100}\vec{V_0}$, where

$$A^{100} = \begin{pmatrix} 0.005 & 0.005 & 0.005 & 0.005 \\ 0.048 & 0.048 & 0.048 & 0.048 \\ 0.085 & 0.085 & 0.085 & 0.085 \\ 0.866 & 0.866 & 0.866 & 0.866 \end{pmatrix}$$

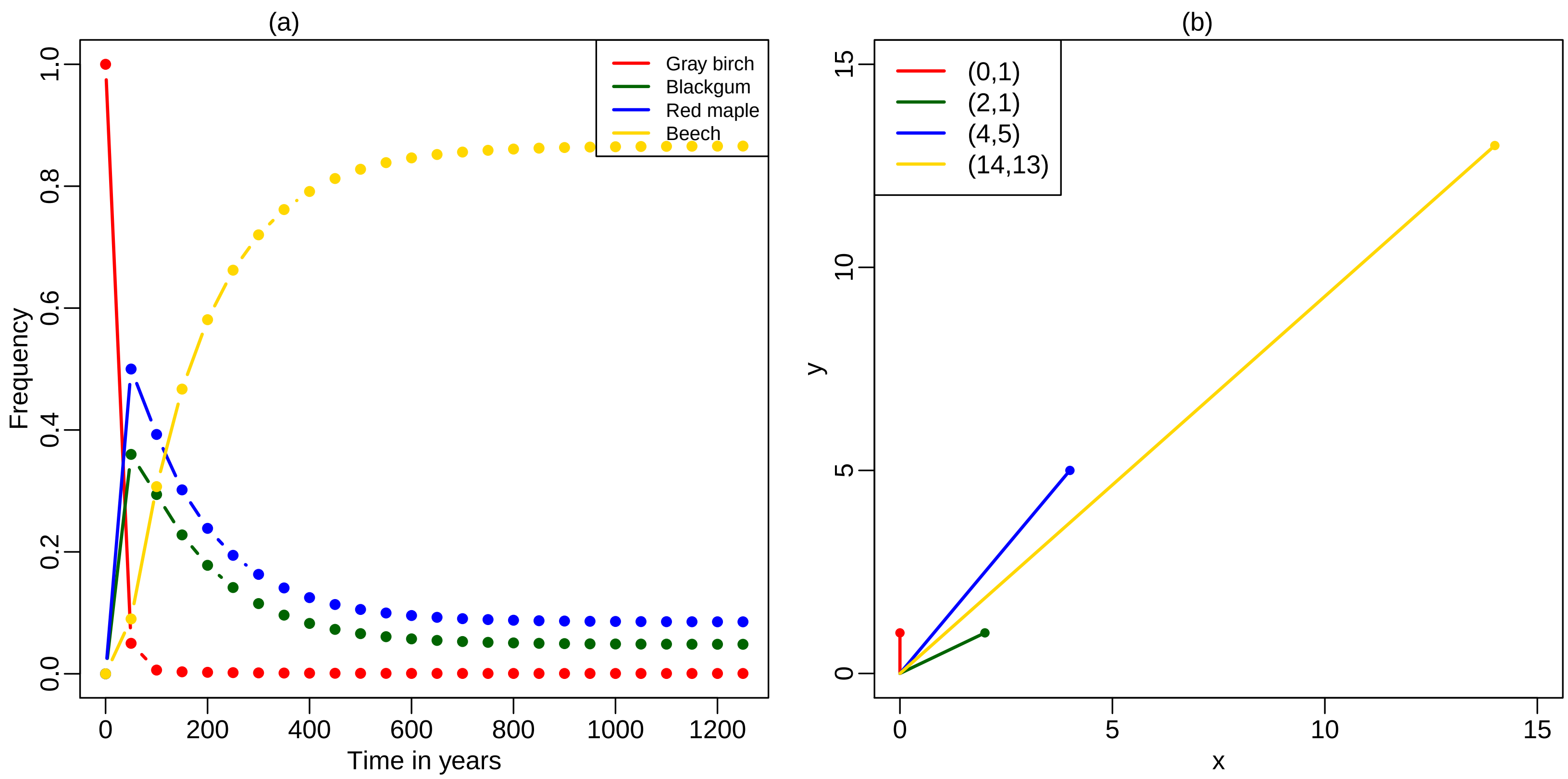
which is a matrix with identical columns.

Now consider an arbitrary vector $\vec{V} = (x\ y\ z\ w)$, where $w = 1 - x - y - z$, and notice that $A^{100}\vec{V} =$

$$\begin{pmatrix} 0.005 & 0.005 & 0.005 & 0.005 \\ 0.048 & 0.048 & 0.048 & 0.048 \\ 0.085 & 0.085 & 0.085 & 0.085 \\ 0.866 & 0.866 & 0.866 & 0.866 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0.005(x+y+z+w) \\ 0.048(x+y+z+w) \\ 0.085(x+y+z+w) \\ 0.866(x+y+z+w) \end{pmatrix} =$$

$(0.005\ 0.048\ 0.085\ 0.866)$, the succession converges into climax state.

This climax vector is an eigenvector of the matrix A!



Left panel the Horn model, right panel transform $(0 \;\; 1)$ with $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, and go on. R-scripts: `horn.R` and `eigen.R`

Eigenvalue problem: $A\vec{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{cases} ax + by = \lambda x \\ cx + dy = \lambda y \end{cases} \quad \text{or} \quad \begin{cases} (a-\lambda)x + by = 0 \\ cx + (d-\lambda)y = 0 \end{cases}$$

Multiply first with $(d-\lambda)$, and second with b:

$$\begin{cases} (d-\lambda)[(a-\lambda)x + by] = 0 \\ b[cx + (d-\lambda)y] = 0 \end{cases}$$

Subtract second from first:

$$[(d-\lambda)(a-\lambda) - bc]x = 0$$

Because $x \neq 0$:

$$(d-\lambda)(a-\lambda) - bc = 0$$

Characteristic equation:

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

Since $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, this can be written as:

$$\lambda^2 - \mathrm{tr}\lambda + \det = 0$$

Hence:

$$\lambda_{1,2} = \frac{\mathrm{tr} \pm \sqrt{\mathrm{tr}^2 - 4\det}}{2}$$

revealing the importance of the trace of a matrix.

Numerical example:

$$A\vec{v} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\mathrm{tr}[A] = 2 \quad \text{and} \quad \det[A] = 1 - 4 = -3$$

Characteristic equation:

$$\lambda_{1,2} = \frac{\mathrm{tr} \pm \sqrt{\mathrm{tr}^2 - 4\det}}{2} = \frac{2 \pm \sqrt{4 + 12}}{2} = \frac{2 \pm 4}{2}$$

Hence:

$$\lambda_1 = 3 \quad \text{and} \quad \lambda_2 = -1$$

Corresponding eigenvectors:

$$\begin{cases} (a-\lambda)x + by = 0 \\ cx + (d-\lambda)y = 0 \end{cases} \quad \text{or} \quad \begin{cases} y = \frac{\lambda-a}{b}\, x \\ x = \frac{\lambda-d}{c}\, y \end{cases}$$

First eigenvector $\lambda_1 = 3, a = 1, b = 2, c = 2, d = 1$:

$$y = \frac{3-1}{2}x = x \quad \text{and} \quad x = \frac{3-1}{2}y = y \quad \text{hence} \quad \vec{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Second eigenvector $\lambda_2 = -1, a = 1, b = 2, c = 2, d = 1$:

$$y = \frac{-1-1}{2}x = -x \quad \text{and} \quad x = \frac{-1-1}{2}y = -y \quad \text{hence} \quad \vec{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

We only need one of the two equations!

Indeed, general case for eigenvectors:

$$\begin{cases} (a-\lambda)x + by = 0 \\ cx + (d-\lambda)y = 0 \end{cases}$$

First equation delivers: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -b \\ a-\lambda \end{pmatrix}$

Indeed, second equals zero (delivers characteristic equation):

$$-bc + (d-\lambda)(a-\lambda) = 0$$

Thus, $\lambda_1 = 3, a = 1, b = 2$: $\vec{v_1} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$ or $\vec{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

for $\lambda_2 = -1, a = 1, b = 2$: $\vec{v_2} = \begin{pmatrix} -2 \\ 1 - -1 \end{pmatrix}$ or $\vec{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Special case, diagonal matrix: $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ or $A = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$

Characteristic equation:

$$(a-\lambda)(d-\lambda) - bc = 0 \qquad \leftrightarrow \qquad (a-\lambda)(d-\lambda) - 0 = 0$$

Hence, the eigenvalues correspond to the diagonal elements:

$$\lambda_1 = a \quad \text{and} \quad \lambda_2 = d$$

Off-diagonal matrix:

$$A = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}: \qquad (0-\lambda)(0-\lambda) - bc = 0 \qquad \leftrightarrow \qquad \lambda_{1,2} = \pm\sqrt{bc}$$