Chapter 5: Killing and Consumption

Density dependence is typically due to another variable of the system

$$R_T = R + cN$$

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{bRN}{h+R} - \delta N = \frac{b(R_T - cN)N}{h + R_T - cN} - \delta N = bN\Big(1 - \frac{h}{h + R_T - cN}\Big) - \delta N$$

Question 5.1

Bacteria in a chemostat: birth rate proportional to consumption *aR*

$$\frac{\mathrm{d}R}{\mathrm{d}t} = s - wR - aRN$$

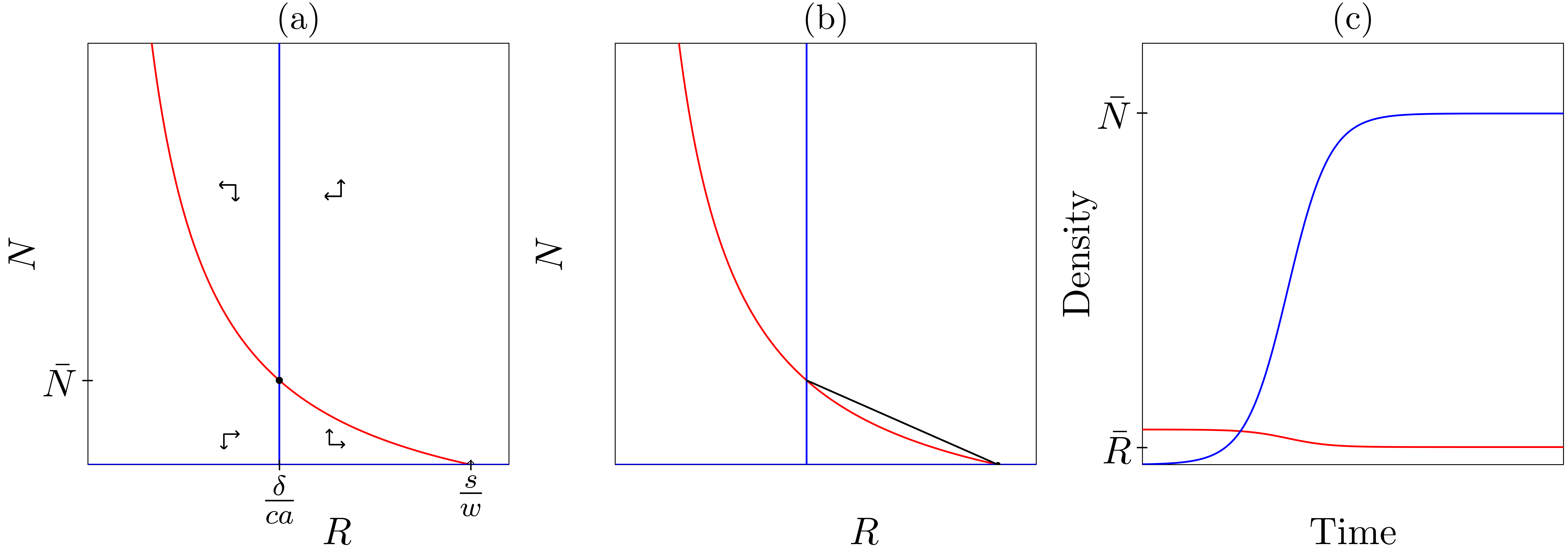
$$\frac{\mathrm{d}N}{\mathrm{d}t} = caRN - (w+d)N = caRN - \delta N$$

$$R = \frac{\delta}{ca}, \quad \bar{N} = \frac{sc}{\delta} - \frac{w}{a}$$

$$R_0 = \frac{ca\bar{R}}{\delta} = \frac{cas}{\delta w}$$

nullclines: *R* '=0: $N = \frac{s}{aR} - \frac{w}{a}$ and: *N* '=0: $N = 0$ or $R = \frac{\delta}{ca}$

Bacteria in a chemostat: birth rate proportional to consumption *aR*



$$\frac{\mathrm{d}R}{\mathrm{d}t} = s - wR - aRN$$

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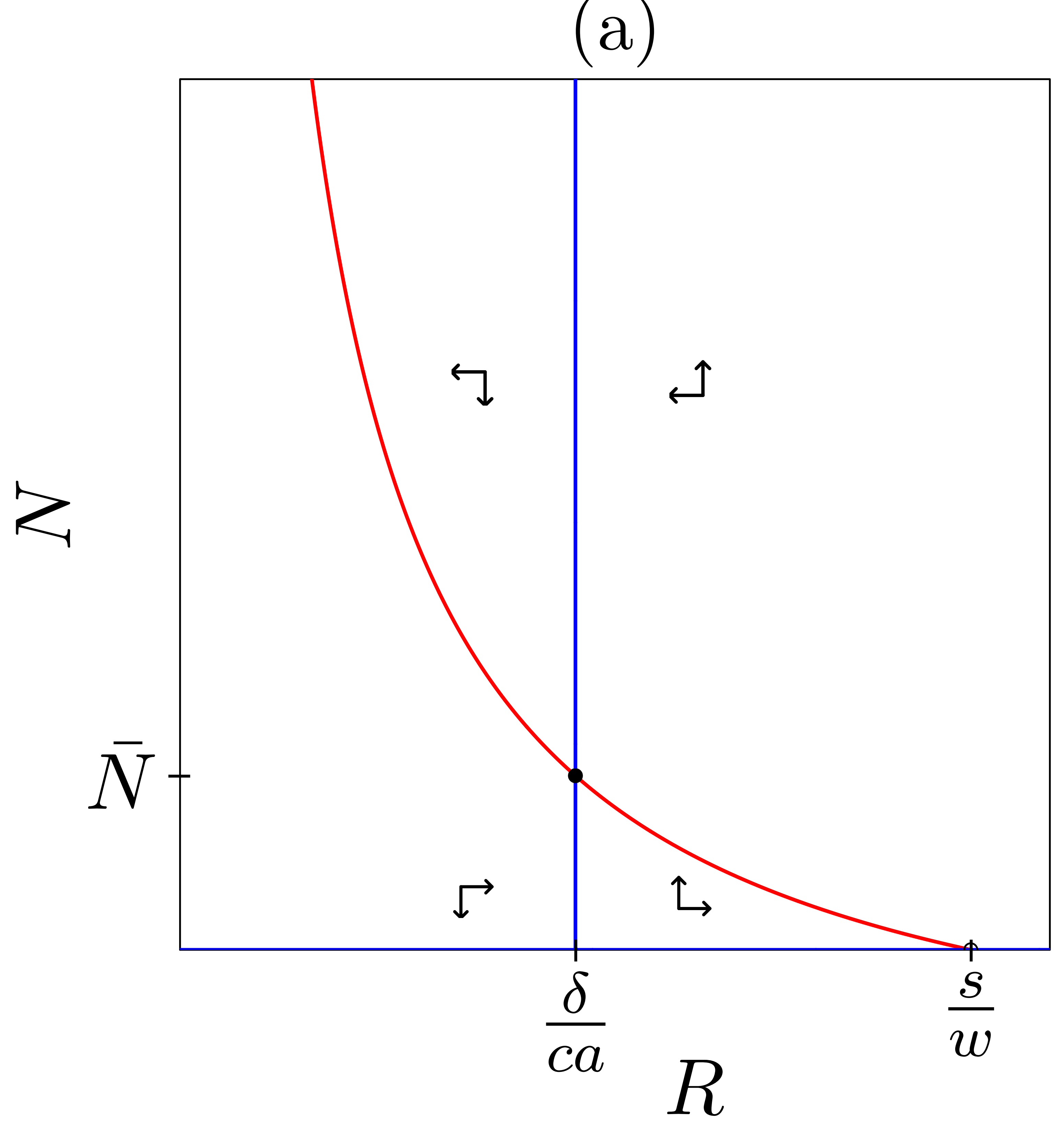
nullclines $N = \frac{s}{aR} - \frac{w}{a}$ $\quad N = 0 \quad \text{or} \quad R = \frac{\delta}{ca}$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = s - wR - aRN$$

$$\frac{\mathrm{d}N}{\mathrm{d}t} = caRN - (w+d)N = caRN - \delta N$$



$$J = \begin{pmatrix} \partial_R f & \partial_N f \\ \partial_R g & \partial_N g \end{pmatrix}_{|(\bar{R},\bar{N})} = \begin{pmatrix} -w - a\bar{N} & -a\bar{R} \\ ca\bar{N} & ca\bar{R} - \delta \end{pmatrix} = \begin{pmatrix} -w - a\bar{N} & -\delta/c \\ ca\bar{N} & 0 \end{pmatrix} = \begin{pmatrix} -\alpha & -\beta \\ +\gamma & 0 \end{pmatrix}$$

$$\mathrm{tr} = -\alpha < 0.$$

$$\det = 0 - -\beta\gamma = \beta\gamma > 0$$

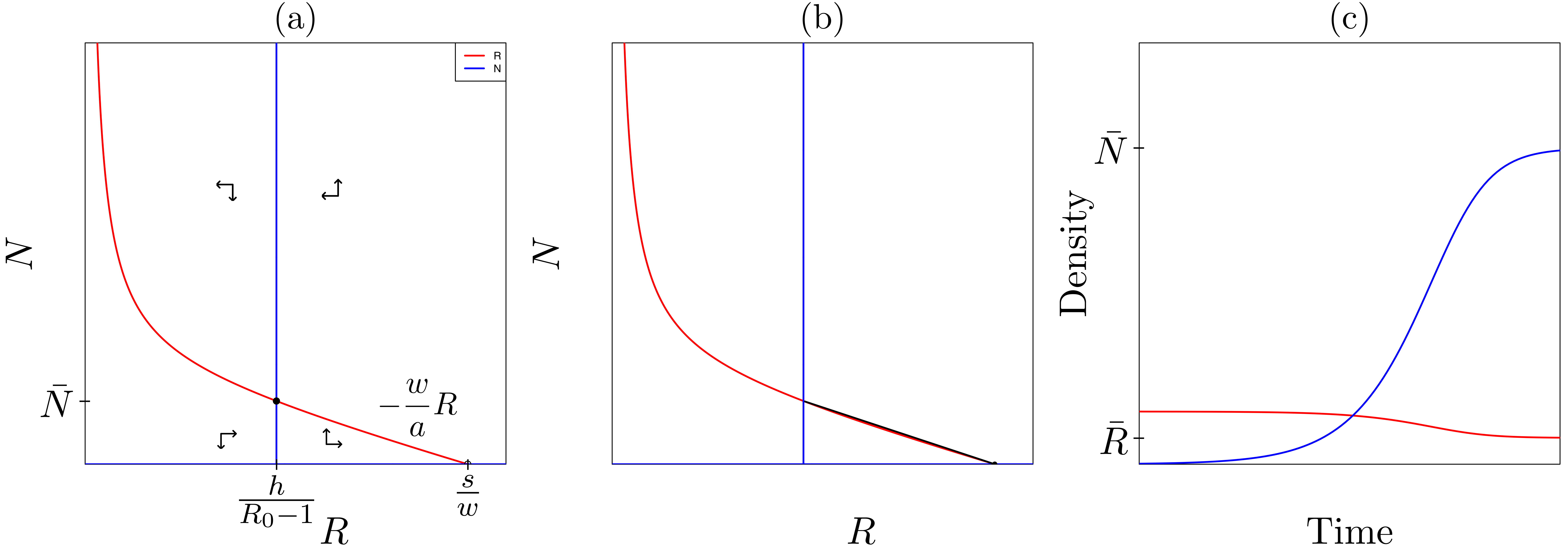
Saturated consumption in a chemostat, birth rate proportional to consumption

$$\frac{\mathrm{d}R}{\mathrm{d}t} = s - wR - \frac{aRN}{h+R} \quad \text{and} \quad \frac{\mathrm{d}N}{\mathrm{d}t} = \frac{caRN}{h+R} - (w+d)N = \frac{caRN}{h+R} - \delta N$$

$$R_0 = \frac{cas}{\delta(wh+s)} \quad \text{or} \quad R_0 = \frac{ca}{\delta}$$

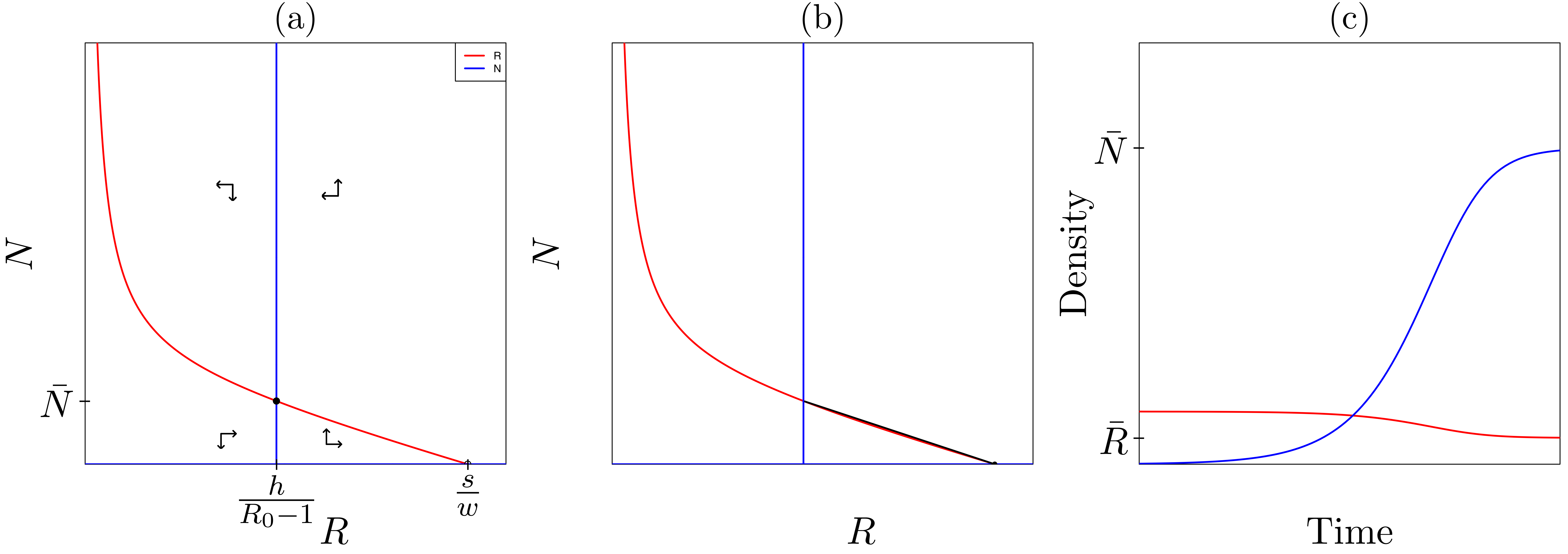
$$\frac{\mathrm{d}N}{\mathrm{d}t} = 0 \ \text{gives} \ \bar{R} = \frac{h\delta}{ca-\delta} = \frac{h}{R_0 - 1}$$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = s - wR - \frac{aRN}{h+R} \quad \text{and} \quad \frac{\mathrm{d}N}{\mathrm{d}t} = \frac{caRN}{h+R} - (w+d)N = \frac{caRN}{h+R} - \delta N$$



$$\frac{\mathrm{d}R}{\mathrm{d}t} = 0 \quad \text{gives} \quad s - wR = \frac{aRN}{h+R} \leftrightarrow N = \frac{(h+R)(s-wR)}{aR} = \frac{h+R}{a}\left(\frac{s}{R} - w\right)$$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = s - wR - \frac{aRN}{h+R} \quad \text{and} \quad \frac{\mathrm{d}N}{\mathrm{d}t} = \frac{caRN}{h+R} - (w+d)N = \frac{caRN}{h+R} - \delta N$$



$$J = \begin{pmatrix} \partial_R f & \partial_N f \\ \partial_R g & \partial_N g \end{pmatrix}_{|(\bar{R},\bar{N})} = \begin{pmatrix} -\alpha & -\beta \\ +\gamma & 0 \end{pmatrix}$$

Replicating resource: Lotka-Volterra model, birth rate proportional to consumption

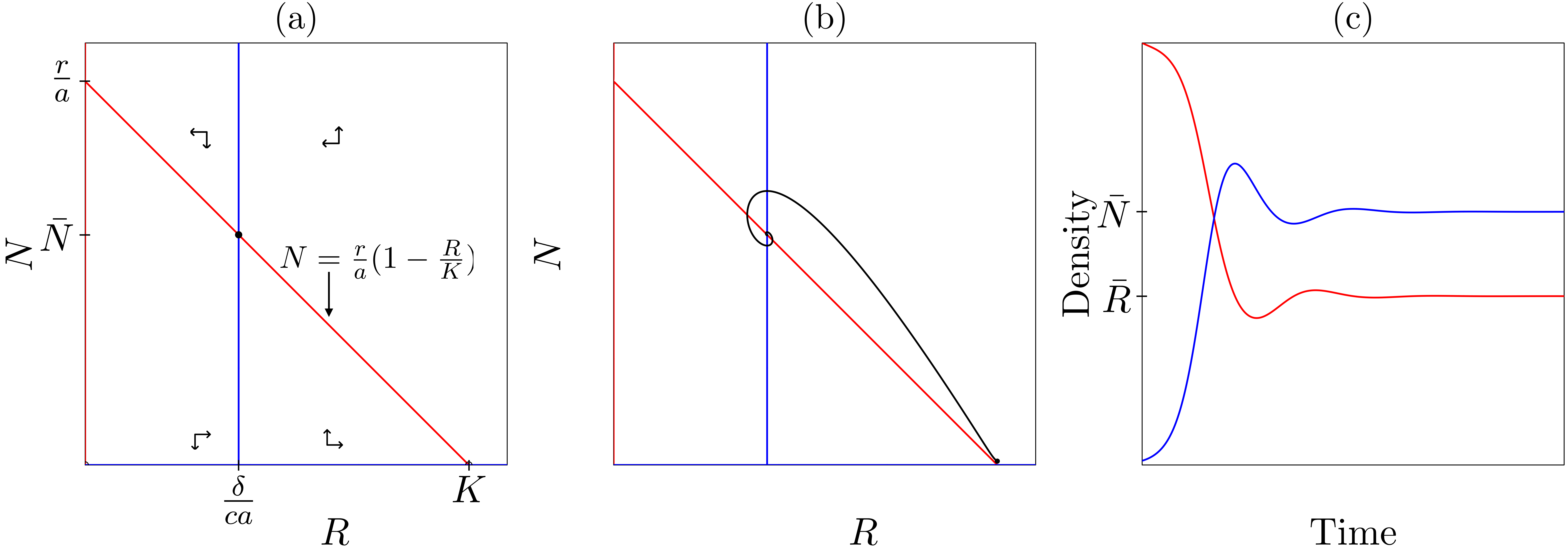
$$\frac{\mathrm{d}R}{\mathrm{d}t} = rR(1 - R/K) - aRN \ , \quad \frac{\mathrm{d}N}{\mathrm{d}t} = caRN - \delta N \ .$$

$\frac{\mathrm{d}N}{\mathrm{d}t} = 0$ gives $\bar{R} = \frac{\delta}{ca}$

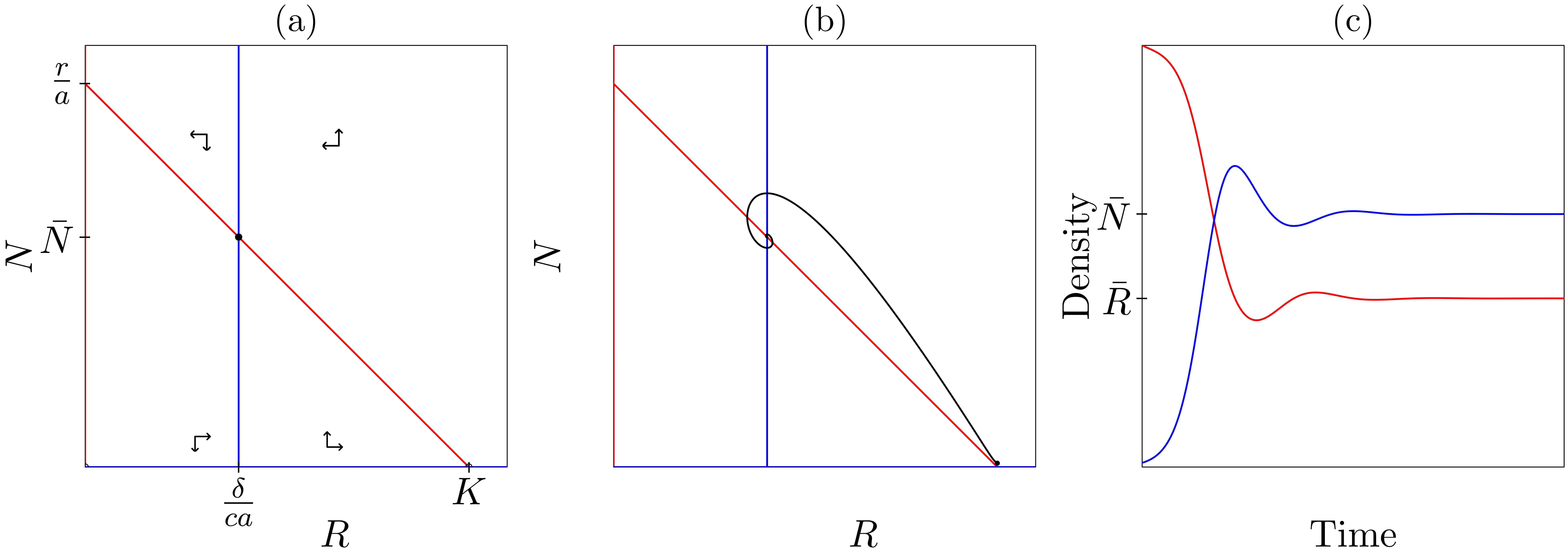
$\frac{\mathrm{d}R}{\mathrm{d}t} = 0$ gives $N = \frac{r}{a}(1 - \frac{R}{K})$

Steady states: $(\bar{R}, \bar{N}) = (0,0), (K,0)$ and $\left(\frac{\delta}{ca}, \frac{r}{a}\left[1 - \frac{\delta}{caK}\right]\right)$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = rR(1 - R/K) - aRN \ , \quad \frac{\mathrm{d}N}{\mathrm{d}t} = caRN - \delta N \ .$$

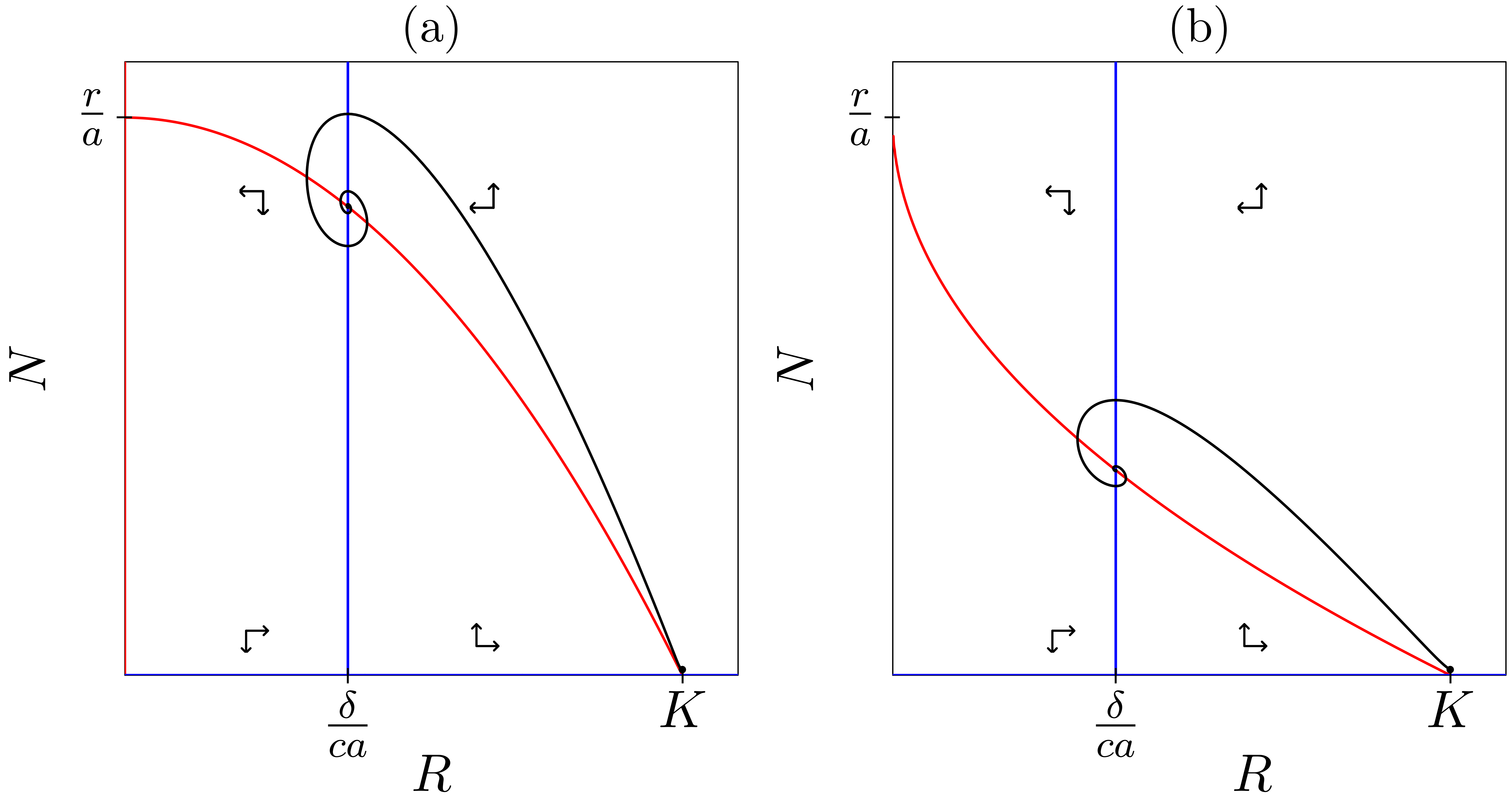


$$\frac{\mathrm{d}R}{\mathrm{d}t} = rR(1-R/K) - aRN \ , \qquad \frac{\mathrm{d}N}{\mathrm{d}t} = caRN - \delta N \ .$$



$$J = \begin{pmatrix} \partial_R f & \partial_N f \\ \partial_R g & \partial_N g \end{pmatrix}_{|(\bar{R},\bar{N})} = \begin{pmatrix} r - \frac{2r}{K}\bar{R} - a\bar{N} & -a\bar{R} \\ ca\bar{N} & ca\bar{R} - \delta \end{pmatrix} = \begin{pmatrix} -\frac{r\delta}{caK} & -\delta/c \\ ca\bar{N} & 0 \end{pmatrix} = \begin{pmatrix} -\alpha & -\beta \\ +\gamma & 0 \end{pmatrix}$$

Generalized Lotka-Volterra model, birth rate proportional to consumption



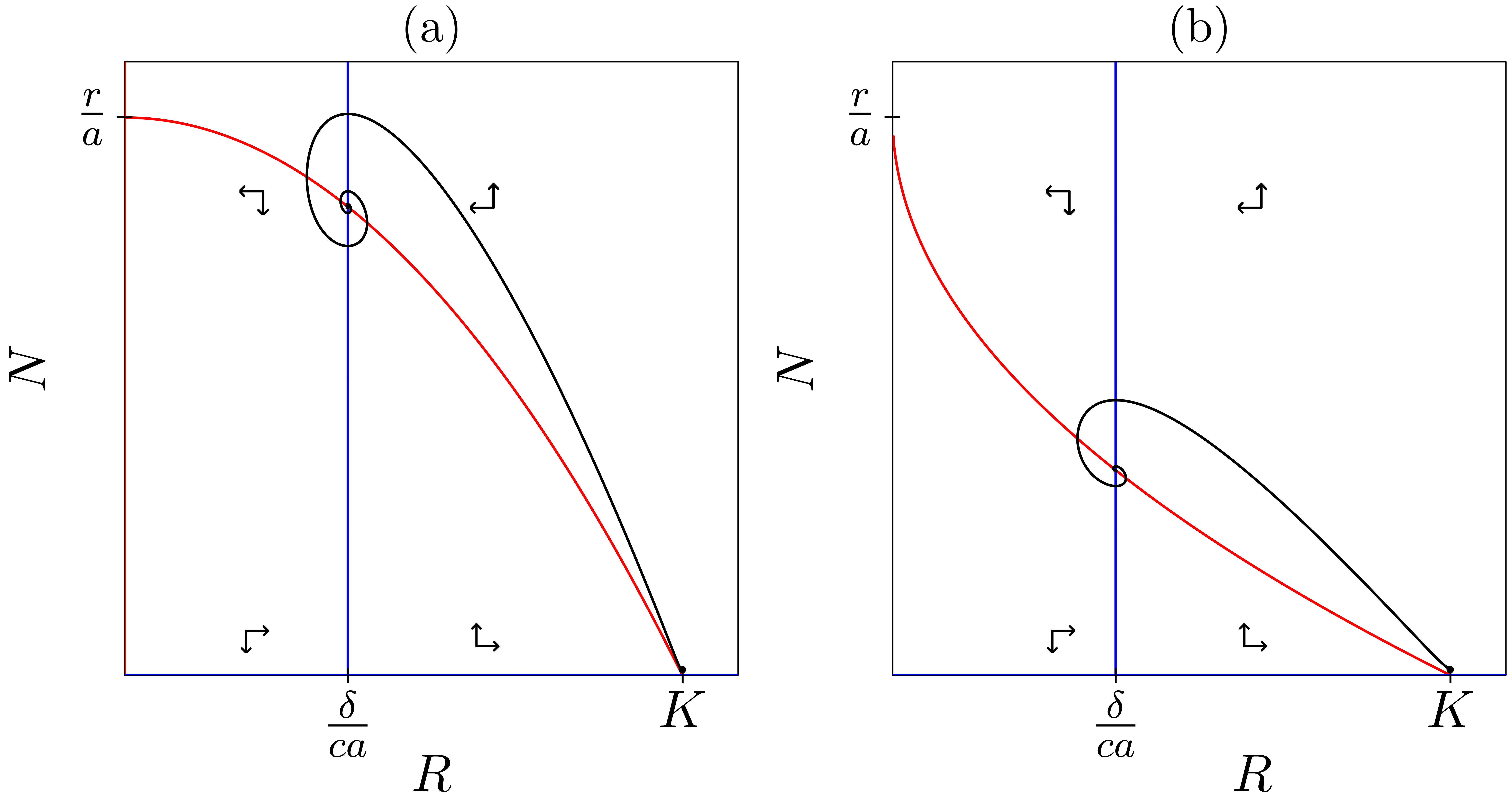
$$\frac{\mathrm{d}R}{\mathrm{d}t} = rR(1 - (R/K)^m) - aRN \;, \quad \frac{\mathrm{d}N}{\mathrm{d}t} = caRN - \delta N$$

$\frac{\mathrm{d}R}{\mathrm{d}t} = 0$ gives $N = \frac{r}{a}(1 - (R/K)^m)$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = [f(R) - aN]\, R$$

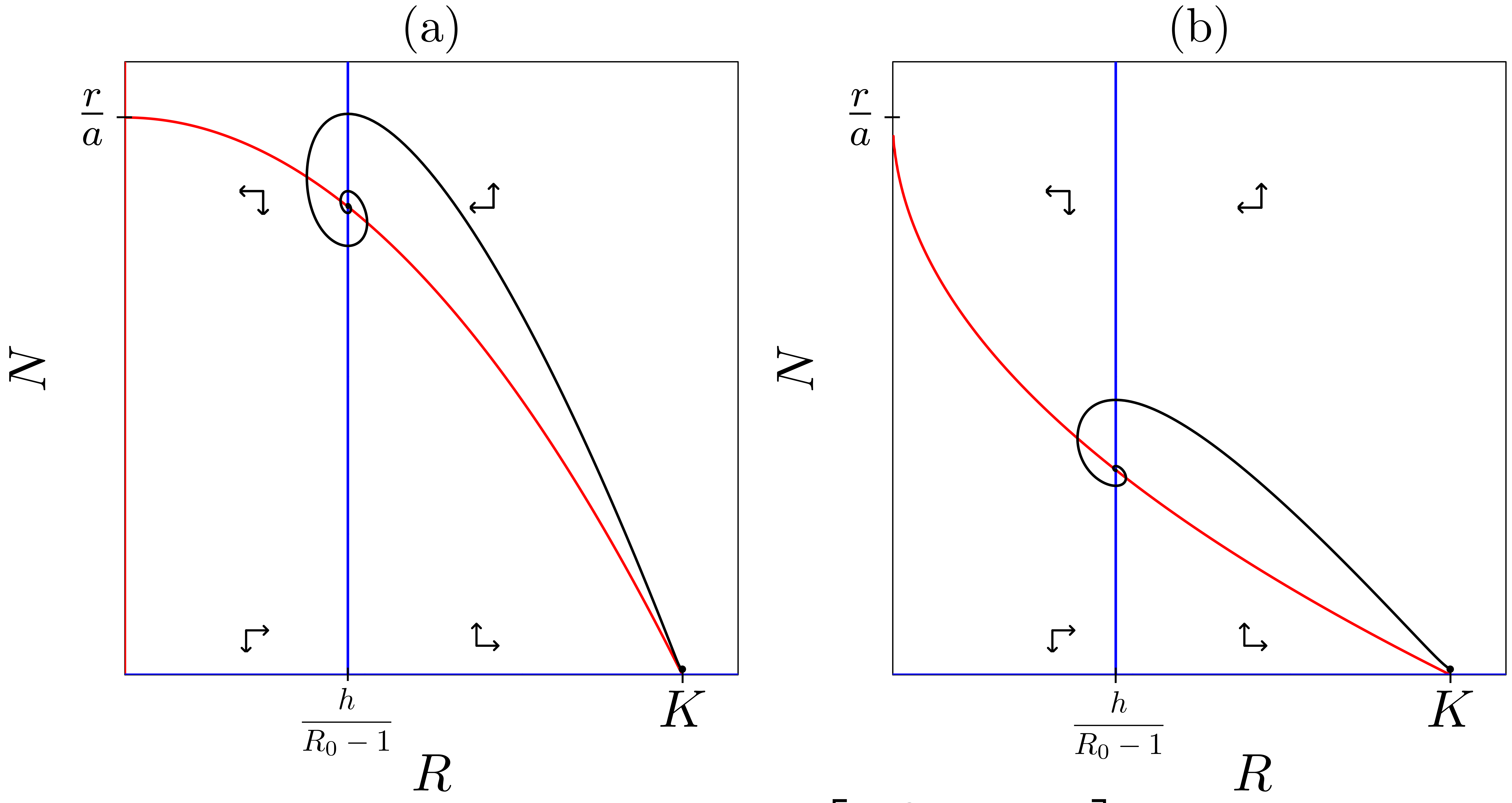
$\frac{\mathrm{d}R}{\mathrm{d}t} = 0$ gives $N = f(R)/c$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = rR(1-(R/K)^m) - aRN\ , \quad \frac{\mathrm{d}N}{\mathrm{d}t} = caRN - \delta N$$



$$J = \begin{pmatrix} \partial_R f & \partial_N f \\ \partial_R g & \partial_N g \end{pmatrix}_{|(\bar{R},\bar{N})} = \begin{pmatrix} -\alpha & -\beta \\ +\gamma & 0 \end{pmatrix}$$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = rR(1-(R/K)^m) - aRN\ , \quad \frac{\mathrm{d}N}{\mathrm{d}t} = caRN - \delta N$$



$$J = \begin{pmatrix} -\alpha & -\beta \\ +\gamma & 0 \end{pmatrix}$$

$$g(aR) = \beta\frac{aR}{H+aR} = \beta\frac{R}{h+R}$$ gives $$\frac{\mathrm{d}N}{\mathrm{d}t} = \left[\frac{\beta R}{h+R} - \delta\right] N = \left[\beta f(R) - \delta\right] N$$

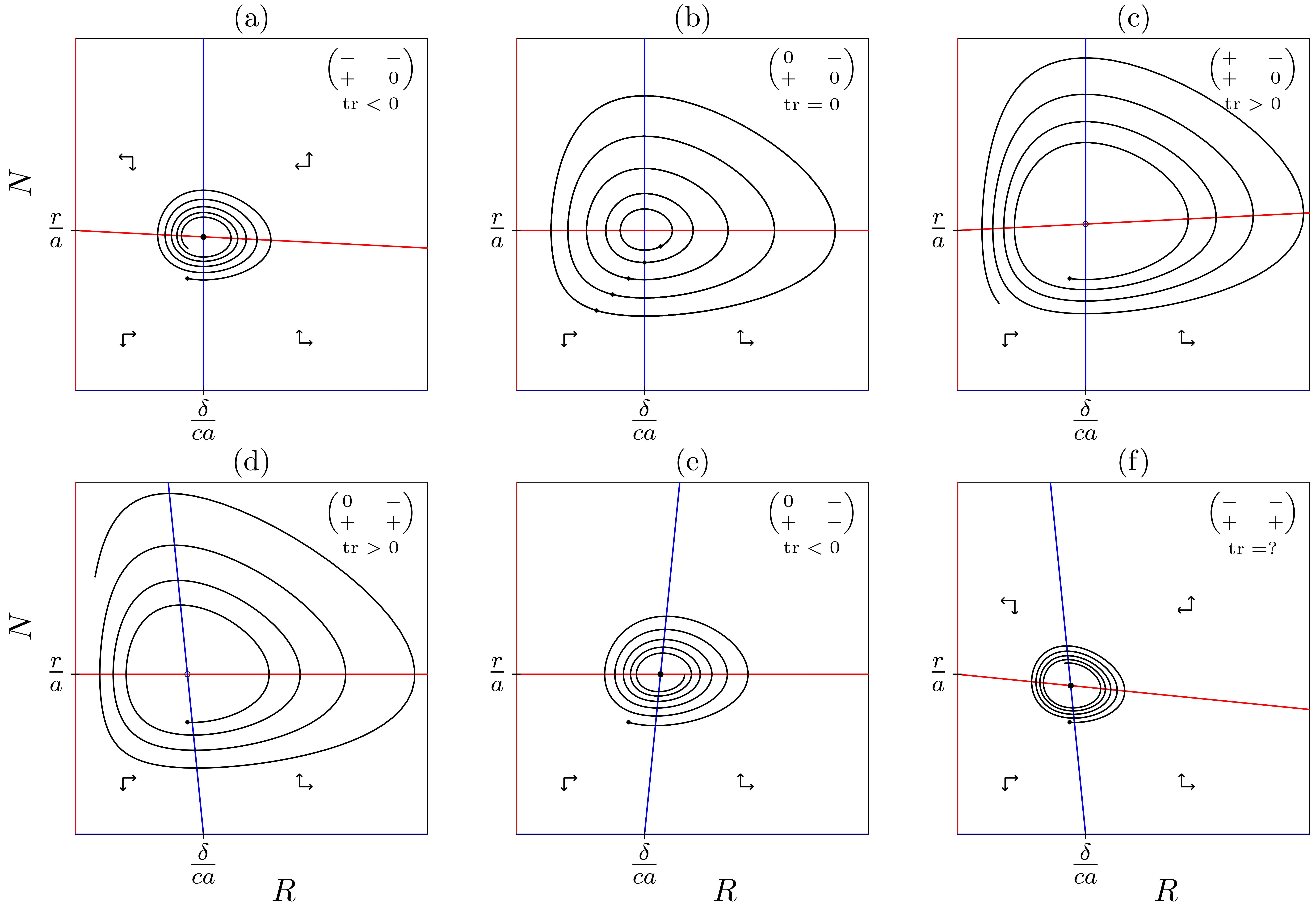
When are horizontal and vertical nullclines a robust result?

$$\frac{\mathrm{d}R}{\mathrm{d}t} = rR - aRN \quad \text{and} \quad \frac{\mathrm{d}N}{\mathrm{d}t} = caRN - \delta N$$

$$N = \frac{r}{a} \quad \text{and} \quad R = \frac{\delta}{ca}$$

$$J = \begin{pmatrix} r - a\bar{N} & -a\bar{R} \\ ca\bar{N} & ca\bar{R} - \delta \end{pmatrix} = \begin{pmatrix} 0 & -\delta/c \\ cr & 0 \end{pmatrix}$$

$$\lambda_{\pm} = \pm\sqrt{-\delta r} = \pm\,\mathrm{i}\sqrt{\delta r}$$


(a)
tr < 0
(b)
tr = 0
(c)
tr > 0
(d)
tr > 0
(e)
tr < 0
(f)
tr =?
N
R
r/a
δ/ca

Enrichment for resource affects consumer steady state only