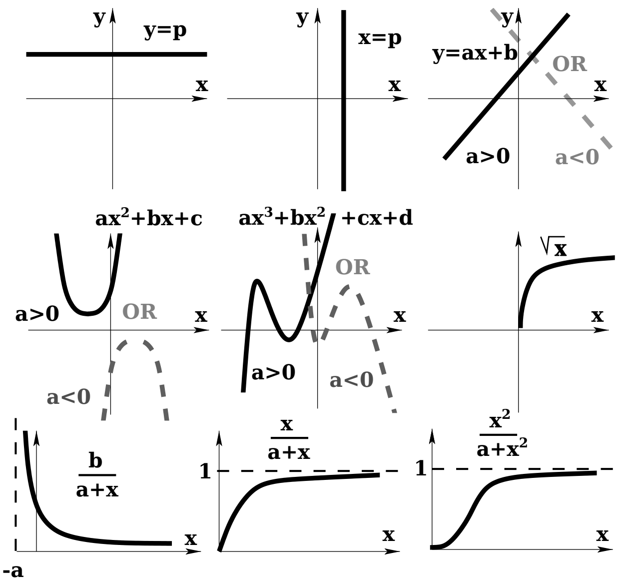
Common graphs:



Quadratic equation: The general solution of a quadratic equation $ax^2 + bx + c = 0$ is given by the so-called *abc-formula*:

$x_{1,2} = \dfrac{-b \pm \sqrt{D}}{2a}$ with $D = b^2 - 4ac$, and **complex numbers** are obtained when $D < 0$, by defining $\mathrm{i}^2 = -1 \Leftrightarrow \mathrm{i} = \sqrt{-1}$, e.g., $\sqrt{-2} = \mathrm{i}\sqrt{2}$.

Linearization:

$f(x,y) \simeq f(\bar{x},\bar{y}) + \partial_x f(\bar{x},\bar{y})\,(x-\bar{x}) + \partial_y f(\bar{x},\bar{y})\,(y-\bar{y})$

The 1D linear differential equation $\mathrm{d}N/\mathrm{d}t = kN$ has the solution: $N(t) = N_0 \mathrm{e}^{kt}$, where N_0 is an (arbitrary) initial value of N.

Eigenvalues and eigenvectors of a 2D matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} ,$$

are defined by the characteristic equation: $\lambda^2 - \mathrm{tr}\lambda + \det = 0$, where $\mathrm{tr} = a + d$ and $\det = ad - bc$, i.e., $\lambda_{1,2} = (\mathrm{tr} \pm \sqrt{D})/2$, where $D = \mathrm{tr}^2 - 4\det$. When $D > 0$ the eigenvalues are real, otherwise they form a complex pair $\lambda_{1,2} = \alpha \pm \mathrm{i}\beta$, where $\alpha = \mathrm{tr}/2$ and $\beta = \sqrt{-D}/2$. The eigenvectors are found by substituting λ_1 and λ_2 into:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -b \\ a - \lambda_i \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d - \lambda_i \\ -c \end{pmatrix}$$

The solution of a linear system of ODEs

$$\begin{cases} \mathrm{d}x/\mathrm{d}t = ax + by \\ \mathrm{d}y/\mathrm{d}t = cx + dy \end{cases} \leftrightarrow \begin{pmatrix} \mathrm{d}x/\mathrm{d}t \\ \mathrm{d}y/\mathrm{d}t \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

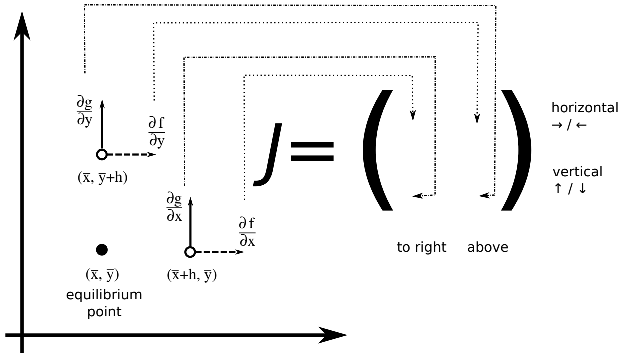
is given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \mathrm{e}^{\lambda_1 t} + C_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \mathrm{e}^{\lambda_2 t} ,$$

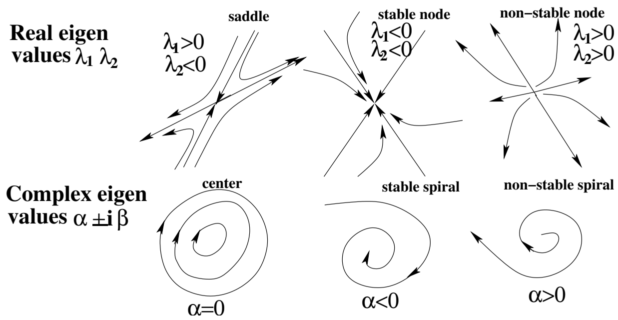
which grows whenever $\lambda_{1,2} > 0$, and where the integration constants C_1 and C_2 define the initial condition $(x(0), y(0))$.

For **general non-linear systems** $\begin{cases} \mathrm{d}x/\mathrm{d}t = f(x,y) \\ \mathrm{d}y/\mathrm{d}t = g(x,y) \end{cases}$ the equilibria are solved from setting $f(x,y) = 0$ and $g(x,y) = 0$. The $x' = 0$ and $y' = 0$ nullclines are given by $f(x,y) = 0$ and $g(x,y) = 0$, respectively. The vector field switches at the nullclines, and can be determined from an extreme value of x and/or y. The equilibrium type can be found by linearizing the ODEs and evaluating the trace and determinant of the **Jacobian** $J = \begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix}$ at the equilibrium.

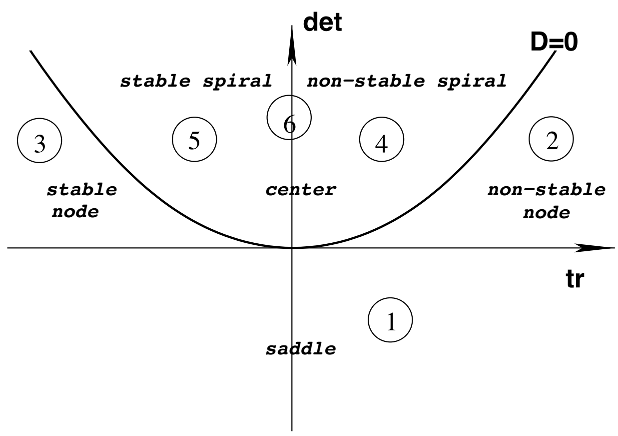
The signs $(+, -, 0)$ of these partial derivatives can be determined using the **graphical Jacobian** method:



Eigenvalues determine the **equilibrium type**, as shown in the figure below, where the straight lines are the eigenvectors:



The equilibrium type can be determined form the trace and determinant of the Jacobian:



Common equations:

Equation	Solution	Conditions
$x^n = p$	$x = p^{\frac{1}{n}} = \sqrt[n]{p}$	$x > 0,\ p > 0$
$g^x = c$	$x = \log_g c$	$x > 0,\ g > 0,\ g \neq 1$
$\log_g x = b$	$x = g^b$	$g > 0,\ g \neq 1$
$e^x = c$	$x = \ln c$	$c > 0$
$\ln x = b$	$x = e^b$	

Working with powers

$a^0 = 1$	$a^{-1} = \frac{1}{a}$	$a^{\frac{1}{2}} = \sqrt{a}$	$a^{\frac{p}{q}} = (\sqrt[q]{a})^p$
$a^1 = a$	$a^{-p} = \frac{1}{a^p}$	$a^{\frac{1}{q}} = \sqrt[q]{a}$	$a^{-\frac{p}{q}} = \frac{1}{(\sqrt[q]{a})^p}$
$0^p = 0$			

$a^p \times a^q = a^{p+q}$	$\frac{a^p}{b^q} = a^p \times b^{-q}$	$(a \times b)^p = a^p \times b^p$
$\frac{a^p}{a^q} = a^p \times a^{-q} = a^{p-q}$	$(a^p)^q = a^{pq}$	$(a^p \times b^q)^r = (a^p)^r \times (b^q)^r = a^{pr} \cdot b^{qr}$

Working with fractions

$\frac{a}{b} = \frac{ca}{cb}$	$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$	$\frac{a}{\frac{b}{c}} = a \times \frac{c}{b} = \frac{ac}{b}$
$\frac{a}{b} \times c = \frac{ca}{b}$	$\frac{ac}{bd} = a \times c \times \frac{1}{b} \times \frac{1}{d} = \frac{a}{b} \times \frac{c}{d}$	$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

Logarithms

The following applies: if $x = n^b$, then $\log_n x = b$, with $n > 0$ and $n \neq 1$. The following rules apply to working with logarithms, provided $a, b, n, q > 0$ and $n, q \neq 1$:

$\log = \log_{10}$	$\log_n ab = \log_n a + \log_n b$	$\log_n a^p = p \times \log_n a$
$\ln = \log_e$	$\log_n \frac{a}{b} = \log_n a - \log_n b$	$\log_n a = \frac{\log_q a}{\log_q n}$

Derivatives

	function	**derivative**
	$g(x) = cf(x)$	$g'(x) = cf'(x)$
sum rule	$p(x) = f(x) + g(x)$	$p'(x) = f'(x) + g'(x)$
product rule	$q(x) = f(x)g(x)$	$q'(x) = f'(x)g(x) + f(x)g'(x).$
chain rule	$r(x) = f(g(x))$	$r'(x) = f'(g(x))g'(x)$
quotient rule	$q(x) = \frac{f(x)}{g(x)}$	$q'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

Derivatives for some common functions:

$x^n \to nx^{n-1}$	$\log_n x \to \frac{1}{x \ln n}$
$e^x \to e^x$	$\sin x \to \cos x$
$g^x \to g^x \ln g$	$\cos x \to -\sin x$
$\ln x \to \frac{1}{x}$	$\tan x \to \frac{1}{\cos^2 x} = 1 + \tan^2 x$

Complex numbers:

The addition of complex numbers is adding their real and imaginary parts, $(a+b\mathrm{i})+(c+d\mathrm{i}) = (a+c+[b+d]\mathrm{i})$, like summing vectors. The multiplication of complex numbers follows similar rules:

$$(a + b\mathrm{i})(c + d\mathrm{i}) = (ac + ad\mathrm{i} + bc\mathrm{i} + bd\mathrm{i}^2) = (ac - bd + [ad + bc]\mathrm{i}).$$

November 10, 2021