Common graphs:

Quadratic equation: The general solution of a quadratic equation $ax^2 + bx + c = 0$ is given by the so-called *abc-formula*:

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} \quad \text{with} \quad D = b^2 - 4ac , \quad \text{and}$$

complex numbers are obtained when $D < 0$, by defining $i^2 = -1 \iff i = \sqrt{-1}$, e.g., $\sqrt{-2} = i\sqrt{2}$.

Linearization:

$$(x, y) \approx f(\bar{x}, \bar{y}) + \partial_x f(\bar{x}, \bar{y})(x-\bar{x}) + \partial_y f(\bar{x}, \bar{y})(y-\bar{y})$$

The 1D linear differential equation $dN/dt = kN$ has the solution: $N(t) = N_0e^{kt}$, where $N_0$ is an (arbitrary) initial value of $N$.

The solution of a linear system of ODEs

$$\begin{cases} dx/dt = ax + by \\ dy/dt = cx + dy \end{cases} \iff \begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

comes from the characteristic equation: $\lambda^2 - tr\lambda + det = 0$, where $tr = a + d$ and $det = ad - bc$, i.e., $\lambda_{1,2} = (tr \pm \sqrt{D})/2$, where $D = tr^2 - 4det$. When $D > 0$ the eigenvalues are real, otherwise they form a complex pair $\lambda_{1,2} = \alpha \pm \beta i$, where $\alpha = tr/2$ and $\beta = \sqrt{-D}/2$. The general solution is given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} e^{\lambda_1 t} + C_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} e^{\lambda_2 t} ,$$

which grows whenever $\lambda_{1,2} > 0$. The eigenvectors are found by substituting $\lambda_1$ and $\lambda_2$ into:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -b \\ a - \lambda_i \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d - \lambda_i \\ -c \end{pmatrix}$$

For general non-linear systems

$$\begin{cases} dx/dt = f(x, y) \\ dy/dt = g(x, y) \end{cases}$$

the equilibria are solved from setting $f(x, y) = 0$ and $g(x, y) = 0$. The $x' = 0$ and $y' = 0$ nullclines are given by $f(x, y) = 0$ and $g(x, y) = 0$, respectively. The vector field switches at the nullclines, and can be determined from an extreme value of $x$ and/or $y$. The equilibrium type can be found by linearizing the ODEs and evaluating the trace and determinant of the Jacobian $J = \begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix}$ at the equilibrium.

The signs $(+, -, 0)$ of these partial derivatives can be determined using the graphical Jacobian method:

Eigenvalues determine the equilibrium type, as shown in the figure below, where the straight lines are the eigenvectors:

The equilibrium type can be determined form the trace and determinant of the Jacobian:
Common equations:

<table>
<thead>
<tr>
<th>Equation</th>
<th>Solution</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^n = p )</td>
<td>( x = p^{\frac{1}{n}} = \sqrt[n]{p} )</td>
<td>( x &gt; 0, \ p &gt; 0 )</td>
</tr>
<tr>
<td>( g^x = c )</td>
<td>( x = \log_g c )</td>
<td>( x &gt; 0, \ g &gt; 0, \ g \neq 1 )</td>
</tr>
<tr>
<td>( \log_g x = b )</td>
<td>( x = g^b )</td>
<td>( g &gt; 0, \ g \neq 1 )</td>
</tr>
<tr>
<td>( e^x = c )</td>
<td>( x = \ln c )</td>
<td>( c &gt; 0 )</td>
</tr>
<tr>
<td>( \ln x = b )</td>
<td>( x = e^b )</td>
<td></td>
</tr>
</tbody>
</table>

Working with powers

\[
\begin{align*}
a^0 &= 1 & a^{-1} &= \frac{1}{a} \\
a^1 &= a & a^{-p} &= \frac{1}{a^p} \\
0^p &= 0 \\
a^p \times a^q &= a^{p+q} \\
\frac{a^p}{a^q} &= a^{p-q} \\
\frac{a^p}{a} &= a^{p-1} \\
\frac{a^p}{a^q} &= \left(\frac{\sqrt{a}}{\sqrt{a}}\right)^p
\end{align*}
\]

\( (a \times b)^p = a^p \times b^p \)

\( (a^p \times b^q)^r = (a^p)^r \times (b^q)^r = a^{pr} \times b^{qr} \)

Working with fractions

\[
\begin{align*}
\frac{a}{b} &= \frac{ca}{cb} \\
\frac{a}{b} \times \frac{c}{d} &= \frac{ac}{bd} \\
\frac{a}{b} \times \frac{c}{d} &= \frac{ac}{bd} \\
\frac{a}{b} + \frac{c}{d} &= \frac{ad+bc}{bd}
\end{align*}
\]

Logarithms

The following applies: if \( x = n^b \), then \( \log_n x = b \), with \( n > 0 \) and \( n \neq 1 \). For instance, \( \log_{10} x \) tells you to what power you should raise 10 (so how many times you should multiply 10 with itself) to get the number \( x \). The following rules apply to working with logarithms, provided \( a, b, n, q > 0 \) and \( n, q \neq 1 \):

\[
\begin{align*}
\log &= \log_{10} \quad & \log_n ab &= \log_n a + \log_n b \quad & \log_n a^p &= p \times \log_n a \\
\ln &= \log_e \quad & \log_n \frac{a}{b} &= \log_n a - \log_n b \quad & \log_n a = \frac{\log_{10} a}{\log_{10} n}
\end{align*}
\]

Derivatives

<table>
<thead>
<tr>
<th>function</th>
<th>derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(x) = cf(x) )</td>
<td>( g'(x) = cf'(x) )</td>
</tr>
<tr>
<td>sum rule</td>
<td>( p(x) = f(x) + g(x) )</td>
</tr>
<tr>
<td>product rule</td>
<td>( q(x) = f(x)g(x) )</td>
</tr>
<tr>
<td>chain rule</td>
<td>( r(x) = f(g(x)) )</td>
</tr>
<tr>
<td>quotient rule</td>
<td>( q(x) = \frac{f(x)}{g(x)} )</td>
</tr>
</tbody>
</table>

Derivatives for some common functions:

\[
\begin{align*}
x^n &\rightarrow nx^{n-1} \\
e^x &\rightarrow e^x \\
g^x &\rightarrow g^x \ln g \\
\ln x &\rightarrow \frac{1}{x}
\end{align*}
\]

Complex numbers:

The addition of complex numbers is adding their real and imaginary parts, \( (a+bi)+(c+di) = (a+c)+(b+d)i \), like summing vectors. The multiplication of complex numbers follows similar rules:

\[ (a+bi)(c+di) = (ac+adi+bci+bdi^2) = (ac-bd)+(ad+bc)i \].